

Surfaces in \mathbb{P}^4 lying on small degree hypersurfaces

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ABSTRACT

Since the work of Ellingsrud and Peskine at the end of 1980s, it has been known that smooth compact complex surfaces in \mathbb{P}^4 with prescribed Chern classes, with the exception of a finite number of families, must lie on hypersurfaces of degree $m \leq 5$. Hence the motivation for the present work: to study smooth surfaces contained in a hypersurface of degree $m \leq 5$ (the meaning of ‘small degree’ in the title). There are two main issues considered in the paper:

- (I) an analogue of the Hartshorne-Lichtenbaum finiteness results for smooth surfaces of general type contained in a small degree hypersurface in \mathbb{P}^4 ,
- (II) a study of the irregularity of smooth surfaces contained in a small degree hypersurface in \mathbb{P}^4 .

For (I) we show that for $m \leq 4$, the number of families is controlled by a function depending on the *ratio* $\alpha = \frac{K^2}{\chi}$ of the Chern invariants (K^2, χ) of surfaces. The same result holds for $m = 5$, with a possible exception of $\alpha = 6$.

For (II) we determine all irregular surfaces contained in a hypersurface of degree $m \leq 3$. We do the same in case $m = 4$, under the additional assumption that a quartic hypersurface has only isolated double points. In general, we show that the Albanese dimension of surfaces contained in quartic hypersurfaces is at most 1.

For $m = 5$, we show that minimal surfaces of Albanese dimension 2 have the irregularity at most 3 and describe the hypothetical surfaces with irregularity 3.

Conceptually, the main idea underlying the above results as well as the whole approach of our paper can be termed as a representation of various geometric and cohomological entities attached to a surface in \mathbb{P}^4 in the category of coherent sheaves on that surface.

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1. INTRODUCTION

Smooth compact complex surfaces in \mathbb{P}^4 constitute an interesting and important part of the study of subvarieties of projective spaces. They are naturally situated at the cross-road of the theories of surfaces, vector bundles, and algebraic cycles. It is well-known that every smooth projective surface can be embedded into \mathbb{P}^5 . To fit into \mathbb{P}^4 , a surface must satisfy an obstruction, known as double point formula,

$$d^2 - 5d - 10(g - 1) + (c_2 - K_X^2) = 0, \quad (1.1)$$

which ties together the degree d , the sectional genus g , and the basic topological invariants (the Chern numbers) K^2 and c_2 of the surface. So one of the basic goals, which is still out of reach, is to find all the surfaces that can be embedded into \mathbb{P}^4 .

Another line of inquiry into the geometry of surfaces in \mathbb{P}^4 has been motivated by a conjecture of Hartshorne and Lichtenbaum stating that rational surfaces in \mathbb{P}^4 form a finite number of families. The work of Ellingsrud and Peskine, [15], solved a more general problem of finiteness of families of surfaces not of general type in \mathbb{P}^4 . A key observation of [15] is that a surface X lying in a hypersurface of degree m in \mathbb{P}^4 has the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ bounded from below by a certain cubic polynomial $P_m(d)$ in the degree d of X . Since then, this result has been greatly improved and clarified. Notably, Decker and Schreyer's work, [9], gives a precise expression for $P_m(d)$,

$$P_m(d) = m \binom{\frac{d}{m} + \frac{m-3}{2}}{3} - \frac{(m-1)^2}{2m} d(d-3) - \binom{m-1}{4} + 1, \quad (1.2)$$

where $m = m_X$ is the smallest degree of a hypersurface in \mathbb{P}^4 containing X ,

$$m_X := \min\{k \in \mathbb{N} \mid h^0(\mathcal{I}_X(k)) > 0\}. \quad (1.3)$$

Then, in [9], it is shown that

$$\chi(\mathcal{O}_X) \geq P_m(d), \quad (1.4)$$

provided $d \geq (m-1)^2 + 2$. This result together with the bound on the sectional genus of [15] implies

THEOREM 1.1 (Ellingsrud and Peskine; Decker and Schreyer). *Given integers χ and $m \geq 2$, the surfaces in \mathbb{P}^4 having holomorphic Euler characteristic χ and lying on a hypersurface of degree m and not on one of a smaller degree form at most a finite number of families.*

Therefore, the “world” of surfaces in \mathbb{P}^4 is governed by the pairs of integers (χ, m) as in Theorem 1.1 and emphasis is placed on the understanding of surfaces that can lie on a hypersurface of given degree. From this point of view, the study of surfaces lying on a small degree hypersurface in \mathbb{P}^4 —*small* meaning $m \in \{2, 3, 4, 5\}$ —would appear, and could perhaps be justified, as a way of obtaining empirical data leading to a better conceptual understanding of surfaces in \mathbb{P}^4 . But in fact, taking the ideas of [15] and [9] a bit further, one can argue that these small degrees really matter precisely for their conceptual significance. To explain this point we fix a pair of integers (K^2, χ) and observe

PROPOSITION 1.2 ([26]). *There exists a number $d(K^2, \chi)$, depending only on K^2 and χ , such that all surfaces in \mathbb{P}^4 with Chern numbers (K^2, χ) and degree $d > d(K^2, \chi)$ lie on a hypersurface of degree ≤ 5 .*

This result tells us that, with a possible exception of finitely many families, the study of surfaces in \mathbb{P}^4 having *prescribed* Chern invariants comes down to understanding surfaces lying on hypersurfaces of degree $m \in \{2, 3, 4, 5\}$. Extrapolating further Proposition 1.2, we suggest

METHA-PRINCIPLE. *An understanding of a property \mathcal{P} for surfaces in \mathbb{P}^4 , with the exception of a finite number of families, comes down to studying the property \mathcal{P} for surfaces contained in hypersurfaces of degree $m \leq 5$.*

After clarifying the origins and motivations for studying surfaces in \mathbb{P}^4 lying on hypersurfaces of small degree let us give an overview of the main results of this paper.

Main results of the paper. There are two main issues considered in this work:

- (I) an analogue of Hartshorne-Lichtenbaum finiteness results for smooth surfaces of general type contained in a small degree hypersurface in \mathbb{P}^4 ,
- (II) a study of the irregularity of smooth surfaces contained in a small degree hypersurface in \mathbb{P}^4 .

We approach (I) as students of the theory of surfaces of general type. To be more precise, let us recall that one of the main problems of that theory is the “geography” problem: characterize the pairs of integers (k, c) which are respectively K_X^2 and $\chi(\mathcal{O}_X)$ of some minimal surface X of general type. The integers $\chi(\mathcal{O}_X)$ and K_X^2 are often referred to as the Chern numbers of X —terminology¹ that we adopt in this paper—and their ratio $\alpha_X := K_X^2/\chi(\mathcal{O}_X)$ (called the slope in the sequel) provides many important dividing lines in this 2-dimensional

¹In view of the Noether formula $12\chi = K_X^2 + c_2$, either (K_X^2, c_2) or (K_X^2, χ) will be referred to as Chern numbers of X .

world of Chern invariants of surfaces of general type. In this notation we often omit the reference to X , if no ambiguity is likely.

It is well known that for a surfaces X of general type $\chi(\mathcal{O}_X) > 0$, so it makes sense to speak about the slope α_X of the Chern numbers even when the surface X is not minimal. This is what we do for surfaces of general type in \mathbb{P}^4 . It turns out that the number of families of such surfaces contained in a small degree hypersurface can be controlled only by the value of the slope α . With the notation (1.3) in mind, we can formulate a sample result concerning part (I).

THEOREM 1.3. *For all surfaces X of general type in \mathbb{P}^4 with $m_X \leq 4$ the following assertions hold.*

- 1) *The slope α of the Chern numbers is smaller than 6.*
- 2) *For every rational number $\alpha < 6$, there exists an integer $d(\alpha)$ such that every surface of slope α has the degree $\leq d(\alpha)$.*
- 3) *For every rational number $\alpha < 6$ and integer $d \leq d(\alpha)$, there exists $\chi(\alpha, d)$ such that every surface of slope α and degree d has the holomorphic Euler characteristic $\leq \chi(\alpha, d)$.*

A similar but somewhat more involved statement holds for surfaces X with $m_X = 5$, see Proposition 4.5. It should be also pointed out that the expressions $d(\alpha)$ and $\chi(\alpha, d)$ in the above theorem are effectively computable. For example, $\chi(\alpha, d)$ is an explicit rational function of α and d .

The above results, in essence, are obtained by a combination of two ingredients: the bound (1.2) of Decker and Schreyer, and inequalities of the form

$$c_2 - K^2 \geq aH \cdot K_X + bd, \quad (1.5)$$

where $\mathcal{O}_X(H)$ is the line bundle embedding X into \mathbb{P}^4 , d is the degree of X , and a and b are positive rational numbers (depending on m_X and explicitly determined in the main body of the paper, see Theorem 3.1, Theorem 4.1 and Theorem 4.2). So our contribution to the Hartshorne-Lichtenbaum problem for surfaces of general type are the inequalities (1.5) above. Of course, one most certainly wonders where those inequalities come from. This and other results of the paper will be explained shortly. For now, let us just say that the existence of these inequalities is an *a priori* consequence of our approach toward the study of surfaces in \mathbb{P}^4 contained in a hypersurface of a small degree.

We now turn to the results concerning the issue (II), the irregularity of smooth surfaces contained in a small degree hypersurface in \mathbb{P}^4 .

THEOREM 1.4. *Let $X \subset \mathbb{P}^4$ be a smooth surface and m_X be the smallest degree of a hypersurface containing X .*

- 1) *If $m_X = 2$, then X is regular.*
- 2) *If $m_X = 3$ and X is irregular, then X is an elliptic scroll of degree $d = 5$. Furthermore, a general cubic hypersurface containing X is a Segre cubic² and the surface X must pass through the ten nodes of every Segre cubic containing it.*
- 3) *If $m_X = 4$, then the Albanese dimension of X is at most 1. Furthermore, if X lies on a quartic hypersurface with only ordinary double points, then X is regular, with a possible exception of X being an elliptic conic bundle of degree $d = -K_X^2 = 8$.*

²Such a cubic has ten nodes, the maximal possible number of nodes for a cubic hypersurface in \mathbb{P}^4 .

- 4) If $m_X = 5$, X is minimal, and its Albanese dimension is 2, then the irregularity of X is 2 or 3.

The above statements illustrate several objectives of our inquiry about the irregularity of surfaces in \mathbb{P}^4 :

- a) determine all irregular surfaces for a given value of m_X ,
- b) for a given m_X , determine all possible values of the Albanese dimension of irregular surfaces,
- c) determine the upper bound on the irregularity for every value of the Albanese dimension that may occur.

Statements 1) and 2) of Theorem 1.4 are examples of a), while 3) and 4) are partial answers to b) and c).

The outline of our approach. In the rest of the introduction we discuss our approach to the study of smooth surfaces contained in a small degree hypersurface in \mathbb{P}^4 . The main idea consists of interpreting the extrinsic datum of a small degree hypersurface as an intrinsic one. This is done first, by thinking of a hypersurface of the minimal degree $m = m_X$ containing a surface $X \subset \mathbb{P}^4$ as a nonzero global section of $\mathcal{N}_X(-K_X - (5 - m)H)$, the normal bundle $\mathcal{N}_X = \mathcal{N}_{X/\mathbb{P}^4}$ tensored with $\mathcal{O}_X(-K_X - (5 - m)H)$, and second, by attaching to that global section a cohomology class, call it ξ , in $H^1(\Theta_X(-K_X - (5 - m)H))$, where Θ_X is the holomorphic tangent bundle of X . The last step is achieved via the coboundary homomorphism

$$H^0(\mathcal{N}_X(-K_X - (5 - m)H)) \longrightarrow H^1(\Theta_X(-K_X - (5 - m)H))$$

coming from the normal exact sequence of $X \subset \mathbb{P}^4$ tensored with $\mathcal{O}_X(-K_X - (5 - m)H)$. Next, via the natural identification $H^1(\Theta_X(-K_X - (5 - m)H)) \cong \text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X - (5 - m)H))$, we interpret the cohomology class ξ as the corresponding extension

$$0 \longrightarrow \mathcal{O}_X(-K_X - (5 - m)H) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0. \quad (1.6)$$

The above extension sequence can be viewed as a reincarnation of a hypersurface of degree m containing X in the category of complexes of coherent sheaves on X . It is clear that it can be viewed as an independent entity and the understanding of its properties constitutes an important part of our approach. Namely, our strategy is to extract as much information as possible from (1.6) independently of the fact that X is embedded into \mathbb{P}^4 and then to use the acquired data to gain an additional insight into the embedding $X \subset \mathbb{P}^4$. As an example, let us take up the question of the (semi)stability of the sheaf \mathcal{T}_ξ . This is something *a priori* independent of the fact that X lies on a hypersurface of degree m in \mathbb{P}^4 , and it gives, by the Bogomolov-Gieseker inequality, the constraint on the Chern invariants of \mathcal{T}_ξ ,

$$3c_2(\mathcal{T}_\xi) \geq c_1^2(\mathcal{T}_\xi).$$

This together with the Chern invariants of \mathcal{T}_ξ , determined from the defining sequence (1.6), provide a prototypical example for the inequalities in (1.5), one of the ingredients to prove Theorem 1.3. Of course, there is no reason for \mathcal{T}_ξ to be semistable. However, even in the unstable case one has a sufficient control of the destabilizing filtration of \mathcal{T}_ξ to recapture the spirit of those inequalities. Though this constitutes a somewhat technical part of our considerations, the main point is quite transparent: the control of the properties of the destabilizing filtration of \mathcal{T}_ξ is enabled by

- the fact that \mathcal{T}_ξ is a part of the extension sequence (1.6) and in particular, that the cotangent sheaf Ω_X is a quotient bundle of \mathcal{T}_ξ , and
- the geometric origin of the extension sequence (1.6) which allows one to relate certain properties of the destabilizing filtration of \mathcal{T}_ξ to the embedding of X in \mathbb{P}^4 .

These remarks indicate that, though the semistable case provides the strongest form of the inequalities (1.5), it is the unstable case that is more interesting. Not only a numerical constraint in the form of (1.5) is obtained, but a certain amount of geometric data encoded in the destabilizing filtration of \mathcal{T}_ξ is gained.

The results of Theorem 1.3 and Proposition 4.5 exploit only the numerical part of the study of (1.6). In this respect, the problem of the irregularity considered in (II) allows to reveal some more geometric aspects of our approach. Namely, the destabilizing subsheaf of \mathcal{T}_ξ , call it \mathcal{G} , is defined as the saturation of the subsheaf of \mathcal{T}_ξ generated by its global sections. These sections are connected to the homomorphism

$$H^0(\mathcal{T}_\xi) \longrightarrow H^0(\Omega_X)$$

induced by the epimorphism in (1.6). It takes no effort to work out conditions for the above homomorphism to be surjective in the case $m \leq 4$. When $m = 5$, we need the hypothesis of minimality for X for our approach to go through. This consideration, for example, immediately implies that for $m \leq 4$ the rank of \mathcal{G} is at most 2 and the proof of the statement 3) of Theorem 1.4 comes down to ruling out the possibility $\text{rank}(\mathcal{G}) = 2$.

The geometric destabilization of \mathcal{T}_ξ just outlined works well as long as the cotangent bundle Ω_X is generically generated by its global sections, *i.e.*, when the Albanese dimension of X is 2. In particular, it gives the result on the possible values of the irregularity in Theorem 1.4 as well as establishes a short list of hypothetical surfaces with irregularity 3, see Theorem 10.9.

However, for surfaces of Albanese dimension 1, the above approach fails. This brings us to the second way of associating an extension sequence to a reduced irreducible hypersurface of degree m containing a surface $X \subset \mathbb{P}^4$. Here again we think of such a hypersurface as a nonzero global section of $\mathcal{N}_X(-K_X - (5 - m)H)$ and then take the Koszul sequence associated to it to obtain the extension

$$0 \longrightarrow \mathcal{O}_X(K_X + (5 - m)H) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{J}_Z(mH) \longrightarrow 0. \quad (1.7)$$

The sequence, in general, is not exact, but let us ignore this for now and assume it is³. Its pertinence to the problem of irregularity of X comes from the identifications

$$H^1(\mathcal{N}_X(K_X)) \cong H^1(\mathcal{N}_X^*)^* \cong H^0(\Omega_X)^*,$$

where the first isomorphism is the Serre duality and the second is a general fact valid for any smooth subvariety of dimension at least 2 in a projective space. From this and (1.7) tensored with $\mathcal{O}_X(K_X)$ it follows that the irregularity of X is controlled by two cohomology groups $H^1(\mathcal{O}_X(2K_X + (5 - m)H))$ and $H^1(\mathcal{J}_Z(K_X + mH))$. For $m \leq 5$, the nonvanishing of the first group gives rather strong restrictions on X . The understanding of the nonvanishing of the second group depends largely on the knowledge of the subscheme Z which could be related to the singular locus of the hypersurface we started with. It is clear that this approach can only work when a good understanding of the singular locus of the hypersurface in question

³The interested reader will find a more detailed discussion of this approach in the introduction of §5.

is available. This is the case for $m = 3$, *i.e.*, for cubic hypersurfaces, when everything can be analyzed completely leading to the elliptic scroll as the only irregular surface with $m = 3$; see Theorem 1.4, 2).

Relation to other works. The subject of surfaces in \mathbb{P}^4 goes back to the classical algebraic geometry, see [30] and the references therein. Most of the results obtained in the subject in the last 30 years are based on the methods of syzygies and of construction of bundles on \mathbb{P}^4 . Our approach of interpreting hypersurfaces containing a surface in \mathbb{P}^4 as certain extensions of sheaves on the surface itself seems to be a relative newcomer in the subject. It was initiated in our previous work [26] with an eye toward the problem of bounding the irregularity of surfaces in \mathbb{P}^4 . Here we enlarge its scope by addressing the Hartshorne-Lichtenbaum problem as well as the problem of classification of irregular surfaces in \mathbb{P}^4 . If for the first problem our contribution is largely tributary to the works [15] and [9], it is with the second problem that we have tried to be as self-contained in our treatment as possible. In particular, in deriving Theorem 1.4, 2), 3), we have avoided to call upon the results on classification of surfaces of small degree in \mathbb{P}^4 . This is motivated (and hopefully justified) by our objective to show/explore various aspects of using the extension constructions to gain an insight into the geometry of surfaces. More importantly, we wanted to see (and show to the reader) how the extension construction ‘pins down’ (hypothetical) irregular surfaces in \mathbb{P}^4 . The proofs of Theorem 1.4, 2) - 4) and Theorem 10.9 provide a substantial evidence that our approach is useful for classifying surfaces in \mathbb{P}^4 .

From the conceptual point of view, our approach could be termed as representing various geometric or cohomological entities by (short exact) complexes of coherent sheaves on a surface in question. The complexes or, better, distinguished triangles in the derived category (of coherent sheaves) we are using turn out to be unstable either for numerical (Bogomolov instability) or for more subtle geometrical reasons. This instability gives rise to a new distinguished triangle which carries more geometry than the initial one. It is in relating the two triangles that one is able to obtain new geometric insights. This is very much in line with more recent developments of methods of derived categories in algebraic geometry such as Bridgeland’s stability conditions, see *e.g.*, [8], [2].

Organization of the paper. In §2, preliminary material is gathered. We start by recalling some facts about the Bogomolov instability and then go on explaining how to relate hypersurfaces in \mathbb{P}^4 containing a surface to the extension sequences of sheaves on that surface. One of the technical results used throughout the paper is Lemma 2.3.

The sections §§3 - 4 are devoted to the Hartshorne-Lichtenbaum problem for surfaces of general type in \mathbb{P}^4 . The main results here are Proposition 3.6 and Proposition 4.5, see also Theorem 1.3 in the introduction.

The rest of the paper is devoted to the problem of the irregularity of surfaces in \mathbb{P}^4 . In §5 we explain how the main ideas of our approach are connected with this problem and in §6 we illustrate some of these ideas in the case of surfaces lying on a quadric hypersurface, see Theorem 6.1.

In §7 the surfaces on a cubic hypersurface are treated. Theorem 7.1 is the main result of this section. §8 is an interlude about elliptic scrolls in \mathbb{P}^4 . The subject is well-known, see [18, 4, 5], but we approach it from the point of view of the (twisted) conormal bundle of a scroll. The main result is Theorem 8.1.

In §9 we treat the case of surfaces on a quartic hypersurfaces with ordinary double points. The main result here is Theorem 9.1.

In §10 the Albanese dimension of surfaces contained in a quartic (rep. quintic) hypersurface is considered: Theorem 10.1 and Theorem 10.9 are the main results of this section.

The Appendix of the paper returns again to the case of an elliptic scroll. The main objective here is to show how the classical configuration of 10 nodes of a Segre cubic hypersurface in \mathbb{P}^4 is related to the geometry of an elliptic scroll contained in it. In particular, we show how the extension construction lifts the famous configuration $(10_4, 15_6)$ of Segre to the category of (short exact complexes of) coherent sheaves on a scroll and we suggest that this should lead to a categorification of $(10_4, 15_6)$ configuration of Segre.

2. NOTATION AND PRELIMINARIES

2.1. Bogomolov instability

Let X be a smooth complex projective surface⁴. We denote by $\text{NS}(X)$ the Néron-Severi group of X . Its rank ρ is called the Picard number of X and the intersection product defines an integral quadratic form on $\text{NS}(X)$, whose real extension to $N(X) := \text{NS}_{\mathbb{R}}(X)$ is of type $(1, \rho - 1)$, by the Hodge Index Theorem. The *positive cone* of X is the open cone

$$N^+(X) = \{D \in N(X) \mid D^2 > 0, H \cdot D > 0, \text{ for some (hence any) ample divisor class } H \text{ on } X\}.$$

Note that $N^+(X)$ contains the ample cone and is contained in the cone of effective divisors.

Let \mathcal{F} be a coherent sheaf on X of rank $r = r_{\mathcal{F}}$. The discriminant of \mathcal{F} is the expression

$$\Delta(\mathcal{F}) = 2r c_2(\mathcal{F}) - (r - 1) c_1^2(\mathcal{F}).$$

A more geometric way to think about $\Delta(\mathcal{F})$ for sheaves of rank $r \geq 1$ is to observe that

$$\frac{\Delta(\mathcal{F})}{2r} = c_2(\mathcal{F}) - \frac{r-1}{2r} c_1^2(\mathcal{F}) = c_2\left(\mathcal{F} \otimes \mathcal{O}_X\left(-\frac{1}{r}c_1(\mathcal{F})\right)\right).$$

The next result is due to Bogomolov and it is used constantly in the sequel.

BOGOMOLOV THEOREM. *Let \mathcal{F} be a torsion free coherent sheaf on a surface X . If $\Delta(\mathcal{F}) < 0$, then there exists a maximal non-trivial saturated subsheaf \mathcal{F}' such that*

- $\Delta(\mathcal{F}') \geq 0$,
- $\frac{c_1(\mathcal{F}')}{r_{\mathcal{F}'}} - \frac{c_1(\mathcal{F})}{r_{\mathcal{F}}} \in N^+(X)$ and $\left(c_1(\mathcal{F}') - \frac{r_{\mathcal{F}'}}{r_{\mathcal{F}}} c_1(\mathcal{F})\right)^2 \geq -\frac{\Delta(\mathcal{F})}{2r_{\mathcal{F}}}.$

In particular, if \mathcal{F} is D -semistable⁵ with respect to an ample divisor D , then $\Delta(\mathcal{F}) \geq 0$.

A torsion free sheaf is called Bogomolov unstable if $\Delta(\mathcal{F}) < 0$ and Bogomolov semistable if $\Delta(\mathcal{F}) \geq 0$. The theorem asserts that a torsion free Bogomolov unstable sheaf contains a maximal Bogomolov semistable subsheaf which destabilizes it with respect to every polarization. Such a subsheaf is called a *maximal Bogomolov destabilizing* subsheaf of the given sheaf.

⁴These hypotheses are assumed throughout the paper whenever we speak about surfaces.

⁵The sheaf \mathcal{F} is D -semistable if $\frac{c_1(\mathcal{F}')}{r_{\mathcal{F}'}} \cdot D \leq \frac{c_1(\mathcal{F})}{r_{\mathcal{F}}} \cdot D$, for any nonzero subsheaf $\mathcal{F}' \subset \mathcal{F}$.

LEMMA 2.1. *Let \mathcal{F} be a locally free sheaf on the surface X . There exists a unique Bogomolov filtration of \mathcal{F} ,*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}$$

such that for each $1 \leq i \leq m$, $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the maximal Bogomolov destabilizing subsheaf of $\mathcal{F}_j/\mathcal{F}_{i-1}$ for every $j > i$.

Proof. One can argue by induction on the rank $r = \text{rank}(\mathcal{F})$. For $r = 1$ the statement is obvious, since by definition locally free sheaves of rank 1 are Bogomolov semistable. So we assume $r \geq 2$ and suppose that the theorem holds for all locally free sheaves of inferior rank. Furthermore, we can assume that \mathcal{F} is Bogomolov unstable (since otherwise there is nothing to prove).

Let \mathcal{F}_1 be a maximal Bogomolov destabilizing subsheaf of \mathcal{F} . By assumption, $\mathcal{F}_1 \neq \mathcal{F}$. Since \mathcal{F}_1 is saturated, the quotient $\mathcal{F}/\mathcal{F}_1$ is torsion free, and therefore \mathcal{F}_1 is reflexive (cf. [20, Proposition 5.22]), hence locally free, since X is a surface. Now, if the quotient $\mathcal{F}/\mathcal{F}_1$ is Bogomolov stable, the filtration reduces to $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{F}$ and we are done. If not, the quotient $\mathcal{F}/\mathcal{F}_1$ has the rank strictly smaller than r and hence the theorem holds for (the reflexive hull or the double dual of) $\mathcal{F}/\mathcal{F}_1$. Hence $(\mathcal{F}/\mathcal{F}_1)^{**}$ admits a unique Bogomolov filtration. Lifting this filtration to \mathcal{F} gives the desired filtration of \mathcal{F} . It is enough to describe the procedure for the lifting of the maximal Bogomolov destabilizing subsheaf, call it \mathcal{G}' , of $\mathcal{F}/\mathcal{F}_1$ and then apply it inductively for other pieces of the Bogomolov filtration of $(\mathcal{F}/\mathcal{F}_1)^{**}$.

Let \mathcal{G}'' be the quotient of the inclusion $\mathcal{G}' \subset \mathcal{F}/\mathcal{F}_1$. We have the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \dashrightarrow & \mathcal{F}_1 & \dashrightarrow & \mathcal{F}_2 & \dashrightarrow & \mathcal{G}' \dashrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/\mathcal{F}_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{G}'' & = & \mathcal{G}'' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where \mathcal{F}_2 is the kernel of the epimorphism $\mathcal{F} \rightarrow \mathcal{G}''$. As before, in this short exact sequence \mathcal{F} is locally free and \mathcal{G}'' is torsion free, hence \mathcal{F}_2 is locally free. Clearly $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{G}' \cong \mathcal{F}_2/\mathcal{F}_1$. We must show that \mathcal{F}_2 is Bogomolov unstable and that \mathcal{F}_1 is a maximal Bogomolov destabilizing subsheaf of \mathcal{F}_2 .

Set $r = \text{rank}(\mathcal{F})$, $r_j = \text{rank}(\mathcal{F}_j)$, and $r_{\mathcal{G}'} = \text{rank}(\mathcal{G}')$. Since

$$\begin{aligned}
\frac{c_1(\mathcal{F}_2)}{r_2} - \frac{c_1(\mathcal{F})}{r} &= \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F})}{r} \right) - \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F}_2)}{r_2} \right) \\
&= \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F})}{r} \right) - \frac{r_{\mathcal{G}'}}{r_2} \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{G}')}{r_{\mathcal{G}'}} \right) \\
&= \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F})}{r} \right) - \frac{r_{\mathcal{G}'}}{r_2} \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F}/\mathcal{F}_1)}{r-r_1} \right) + \frac{r_{\mathcal{G}'}}{r_2} \left(\frac{c_1(\mathcal{G}')}{r_{\mathcal{G}'}} - \frac{c_1(\mathcal{F}/\mathcal{F}_1)}{r-r_1} \right) \\
&= \left(1 - \frac{r_{\mathcal{G}'}}{r_2} \frac{r}{r-r_1} \right) \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F})}{r} \right) + \frac{r_{\mathcal{G}'}}{r_2} \left(\frac{c_1(\mathcal{G}')}{r_{\mathcal{G}'}} - \frac{c_1(\mathcal{F}/\mathcal{F}_1)}{r-r_1} \right) \\
&= \frac{r(r-r_2)}{r(r-r_1)} \left(\frac{c_1(\mathcal{F}_1)}{r_1} - \frac{c_1(\mathcal{F})}{r} \right) + \frac{r_{\mathcal{G}'}}{r_2} \left(\frac{c_1(\mathcal{G}')}{r_{\mathcal{G}'}} - \frac{c_1(\mathcal{F}/\mathcal{F}_1)}{r-r_1} \right),
\end{aligned}$$

we see that $c_1(\mathcal{F}_2)/r_2 - c_1(\mathcal{F})/r \in N^+(X)$. Hence $\Delta(\mathcal{F}_2) < 0$, since otherwise \mathcal{F}_2 would be a Bogomolov destabilizing subsheaf of \mathcal{F} and this contradicts the maximality of \mathcal{F}_1 .

Thus we now have constructed a Bogomolov unstable subsheaf \mathcal{F}_2 of \mathcal{F} and we claim that \mathcal{F}_1 is its maximal Bogomolov destabilizing subsheaf. Indeed, if \mathcal{F}_1 is not a maximal Bogomolov destabilizing subsheaf of \mathcal{F}_2 , then there exists a Bogomolov semistable (locally free) subsheaf \mathcal{F}' such that $\mathcal{F}_1 \subset \mathcal{F}' \subset \mathcal{F}_2$ and such that $c_1(\mathcal{F}')/r' - c_1(\mathcal{F}_2)/r_2 \in N^+(X)$. But then, by the previous argument,

$$\frac{c_1(\mathcal{F}')}{r'} - \frac{c_1(\mathcal{F})}{r} = \left(\frac{c_1(\mathcal{F}')}{r'} - \frac{c_1(\mathcal{F}_2)}{r_2} \right) + \left(\frac{c_1(\mathcal{F}_2)}{r_2} - \frac{c_1(\mathcal{F})}{r} \right) \in N^+(X),$$

contradicting the maximality of \mathcal{F}_1 . \square

2.2. From hypersurfaces of small degree to extension classes

Let $X \subset \mathbb{P}^4$ be a smooth surface. In what follows we denote by $\mathcal{N}_X = \mathcal{N}_{X/\mathbb{P}^4}$ the normal bundle of X in \mathbb{P}^4 and by H a hyperplane section of X . The normal bundle \mathcal{N}_X is of rank 2 on X with determinant $\det(\mathcal{N}_X) = \wedge^2 \mathcal{N}_X = \mathcal{O}_X(K_X + 5H)$. The conormal bundle satisfies

$$\mathcal{J}_X/\mathcal{J}_X^2 = \mathcal{N}_X^* \cong \det(\mathcal{N}_X^*) \otimes \mathcal{N}_X = \mathcal{N}_X(-K_X - 5H), \quad (2.1)$$

where \mathcal{J}_X is the ideal sheaf of X in \mathbb{P}^4 and the second identification comes from the fact that the rank of \mathcal{N}_X^* is 2.

We now assume that X lies on a hypersurface of degree $m \leq 4$ and not on one of a smaller degree, *i.e.*, $m = m_X$. From the first equality in (2.1), it follows that $H^0(\mathcal{N}_X^*(mH))$ does not vanish. Hence, by two other identifications in (2.1), we obtain

$$H^0(\mathcal{N}_X(-K_X - (5-m)H)) = H^0(\mathcal{N}_X^*(mH)) \neq 0. \quad (2.2)$$

Set $t = 5 - m$ and observe that $t \in \{1, 2, 3\}$. We wish to interpret nonzero sections of $\mathcal{N}_X(-K_X - tH)$ cohomologically. For this, consider the normal sequence of X in \mathbb{P}^4 tensored with $\mathcal{O}_X(-K_X - tH)$,

$$0 \longrightarrow \Theta_X(-K_X - tH) \longrightarrow \Theta_{\mathbb{P}^4}|_X(-K_X - tH) \longrightarrow \mathcal{N}_X(-K_X - tH) \longrightarrow 0.$$

This implies that $H^0(\mathcal{N}_X(-K_X - tH))$ fits into the following exact sequence of cohomology groups

$$H^0(\Theta_{\mathbb{P}^4}|_X(-K_X - tH)) \longrightarrow H^0(\mathcal{N}_X(-K_X - tH)) \xrightarrow{\delta_X} H^1(\Theta_X(-K_X - tH)). \quad (2.3)$$

The following result improves a part of Lemma 5.3 in [26].

LEMMA 2.2. *Let s be a nonzero global section of $\mathcal{N}_X(-K_X - tH)$, $1 \leq t \leq 3$, corresponding to a 3-fold of degree $m = 5 - t$ containing X . If the Kodaira dimension of X is non-negative, then the cohomology class $\delta_X(s) \neq 0$.*

Proof. We only prove the case $t = 1$, i.e., X is contained in a quartic hypersurface, since the other cases are much easier.

Assume $\delta_X(s) = 0$ in $H^1(\Theta_X(-K_X - H))$, then s is the image of a nonzero global section \tilde{s} of $\Theta_{\mathbb{P}^4}|_X(-K_X - H)$. From the Euler sequence of $\Theta_{\mathbb{P}^4}$, we deduce that either $H^0(\mathcal{O}_X(-K_X)) \neq 0$ or $\ker(H^1(-K_X - H) \rightarrow H^0(H)^* \otimes H^1(-K_X)) \neq 0$. In both cases we see that $H^0(\mathcal{O}_C(-K_X)) \neq 0$ for every $C \in |H|$. Hence $H \cdot (-K_X) \geq 0$.

If $H \cdot (-K_X) > 0$, then $h^0(\mathcal{O}_X(mK_X)) = 0$ for every positive integer m , hence X rational or rationally ruled. If $H \cdot (-K_X) = 0$, then $\mathcal{O}_C(-K_X) = \mathcal{O}_C$, for every $C \in |H|$. This tells us that $H^0(\mathcal{O}_X(-K_X)) \neq 0$ and hence $K_X = 0$. Therefore, X is minimal and it is either a K3 or an abelian surface.

If X is a minimal K3 surface, then the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_X(H)) \longrightarrow H^0(\mathcal{O}_C(H)) \longrightarrow 0,$$

with C a general curve in $|H|$, implies that $g(C) = h^0(\mathcal{O}_C(H)) = h^0(\mathcal{O}_X(H)) - 1 = 4$. Hence $d = H^2 = 2g(C) - 2 = 6$. But then the sequence

$$0 \longrightarrow \mathcal{J}_X(2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(2) \longrightarrow \mathcal{O}_X(2H) \longrightarrow 0$$

implies

$$h^0(\mathcal{J}_X(2)) \geq h^0(\mathcal{O}_{\mathbb{P}^4}(2)) - h^0(\mathcal{O}_X(2H)) = \binom{6}{2} - \frac{(2H)^2}{2} - 2 = 1,$$

i.e., there is a quadric passing through X and this is contrary to our assumption.

Thus we are left with the second case: X a minimal abelian surface. From the double point formula (1.1) it follows that $d = 10$. We show that a minimal abelian surface of degree 10 can not lie on a hypersurface of degree 4. This follows from the following observation.

Claim. A quartic hypersurface $Q \in \mathbb{P}^4$ containing a minimal abelian surface X must be a cone over a quartic surface $S \subset \mathbb{P}^3$ with at most isolated singularities.

Let us assume the claim and derive a contradiction. We view \mathbb{P}^4 as $\mathbb{P}(H^0(\mathcal{O}_X(H))^*)$ and denote by $[v]$, $v \in H^0(\mathcal{O}_X(H))^*$, the vertex of Q . Under the projection from $[v]$ the surface X becomes a finite covering of a quartic surface S lying in some \mathbb{P}^3 , complementary to $[v]$. Let m be the degree of this covering. It is related to the intersection of X with a ruling l of the cone Q as follows.

$$(X \cdot l)_Q = \begin{cases} m & \text{if } [v] \notin X \\ m - 1 & \text{if } [v] \in X. \end{cases}$$

But for a general plane Λ passing through $[v]$,

$$10 = d = (X \cdot \Lambda)_{\mathbb{P}^4} = \begin{cases} 4m & \text{if } [v] \notin X \\ 4(m - 1) + 1 & \text{if } [v] \in X \end{cases}$$

and neither case is possible.

We now turn to the proof of the claim. Let us recall the situation: s is a nonzero global section in $\mathcal{N}_X(-H)$ corresponding to a quartic hypersurface Q containing X , and it is the image of a global section \tilde{s} of $\Theta_{\mathbb{P}^4}|_X(-H)$. The argument is divided into two steps.

Step 1. We claim that the scheme of zeros $Z_s = \{s = 0\}$ of the global section s is 0-dimensional. Indeed, let us assume that this is not the case and let Γ be a reduced, irreducible curve in Z_s . This means that $s = \gamma s'$, where $\gamma \in H^0(\mathcal{O}_X(\Gamma))$ is a global section defining Γ and $s' \in H^0(\mathcal{N}_X(-H - \Gamma))$. From the commutative diagram

$$\begin{array}{ccccc} H^0(\Theta_{\mathbb{P}^4}|_X(-H - \Gamma)) & \longrightarrow & H^0(\mathcal{N}_X(-H - \Gamma)) & \longrightarrow & H^1(\Theta_X(-H - \Gamma)) \\ \gamma \cdot \downarrow & & \downarrow \gamma \cdot & & \\ H^0(\Theta_{\mathbb{P}^4}|_X(-H)) & \longrightarrow & H^0(\mathcal{N}_X(-H)) & & \end{array}$$

it follows that either $H^0(\Theta_{\mathbb{P}^4}|_X(-H - \Gamma)) \neq 0$ or $H^1(\Theta_X(-H - \Gamma)) \neq 0$. The latter possibility leads to $H^1(\mathcal{O}_X(-H - \Gamma)) \neq 0$, since $H^1(\Theta_X(-H - \Gamma)) \cong \oplus H^1(\mathcal{O}_X(-H - \Gamma))$. But then, $H^0(\mathcal{O}_\Gamma(-H)) \cong H^1(\mathcal{O}_X(-H - \Gamma)) \neq 0$ which is impossible. The former possibility leads, using the Euler sequence for $\Theta_{\mathbb{P}^4}$, to

$$\ker(H^1(\mathcal{O}_X(-H - \Gamma)) \rightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(-\Gamma))) \neq 0,$$

which means that $H^0(\mathcal{O}_C(-\Gamma)) \neq 0$ for every $C \in |H|$, which is not possible either.

Step 2. Q is a cone over a quartic surface. Indeed, on the one hand we consider the diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_X(H))^* & & \\ \downarrow & \searrow & \\ H^0(\Theta_{\mathbb{P}^4}|_X(-H)) & \longrightarrow & H^0(\mathcal{N}_X(-H)) \end{array} \quad (2.4)$$

where the horizontal arrow comes from the normal sequence of X in \mathbb{P}^4 and the vertical one is part of the Euler sequence of $\Theta_{\mathbb{P}^4}$ (tensored with $\mathcal{O}_{\mathbb{P}^4}(-1)$) restricted to X . Both maps are isomorphisms. This implies that the section s is the image of a unique element $v \in H^0(\mathcal{O}_X(H))^*$.

On the other hand, we have the Koszul sequence associated to s ,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{N}_X(-H) \xrightarrow{s \wedge} \mathcal{J}_{Z_s}(3H) \longrightarrow 0.$$

which is exact by *Step 1*. Combining this with the slanted arrow in (2.4) gives the following diagram

$$\begin{array}{ccccccc} & & H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X & & & & \\ & & \downarrow e & \searrow & & & \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & \mathcal{N}_X(-H) & \xrightarrow{s \wedge} & \mathcal{J}_{Z_s}(3H) \longrightarrow 0 \end{array} \quad (2.5)$$

Furthermore, the homomorphism

$$\partial : H^0(\mathcal{O}_X(H))^* \longrightarrow H^0(\mathcal{J}_{Z_s}(3H)) \quad (2.6)$$

induced from the above diagram on the level of global sections is the partial differentiation. Namely, let F be a homogeneous polynomial defining Q , then the homomorphism ∂ is given by

$$H^0(\mathcal{O}_X(H))^* \ni u \mapsto \partial(u) = \partial_u F|_X \in H^0(\mathcal{J}_{Z_s}(3H)).$$

By construction $\partial(v)$ factors via $H^0(\mathcal{N}_X(-H))$, *i.e.*, we have

$$\partial_v F|_X = \partial(v) = s \wedge (e(v)) = s \wedge s = 0.$$

This means that the homogeneous polynomial of degree 3, $\partial_v F \in \text{Sym}^3 H^0(\mathcal{O}_X(H))$, vanishes on X . But since the surface X is not contained in any hypersurface of degree less than 4, we conclude that $\partial_v F = 0$. Equivalently, $F \in \text{Sym}^4(\ker(v))$, *i.e.*, the 3-fold of degree four Q is the cone in $\mathbb{P}(H^0(\mathcal{O}_X(H))^*)$ with vertex $[v]$ and base the quartic surface defined by F in $\mathbb{P}^3 = \mathbb{P}(\ker(v)^*)$.

The last assertion, stating that the quartic surface S defined by F in $\mathbb{P}(\ker(v)^*)$ has at most isolated singularities, follows from the observation that a curve, call it Γ , in the singular locus of S produces a surface Σ in Q —the cone over Γ with vertex at $[v]$ —and this surface lies in the singular locus of the quartic Q . But then the surfaces X intersects Σ along a curve which is part of the zero-locus of the section s . This contradicts *Step 1*. \square

From now on we assume that the cohomology class $\delta_X(s) \in H^1(\Theta_X(-K_X - tH))$ in Lemma 2.2 is nonzero. The identification

$$H^1(\Theta_X(-K_X - tH)) \cong \text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X - tH))$$

allows us to interpret a cohomology class on the left as an extension sequence of sheaves on X . The following result constitutes one of the main technical ingredients of this study.

LEMMA 2.3. *Let X be a smooth projective surface and let M be a divisor on X . Let $\xi \in H^1(\Theta_X(-K_X - M))$ be a nonzero cohomology class and let*

$$0 \longrightarrow \mathcal{O}_X(-K_X - M) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0. \quad (2.7)$$

be the corresponding extension sequence. Assume that \mathcal{T}_ξ contains a subsheaf \mathcal{F} of rank 2 such that the induced morphism $\mathcal{F} \rightarrow \Omega_X$ is generically an isomorphism. Then the following holds.

- 1) *The canonical divisor of X decomposes as $K_X = L + E$, where $L = c_1(\mathcal{F})$ and E is the support of the cokernel $\text{coker}(\mathcal{F} \rightarrow \Omega_X)$, a nonzero effective divisor on X .*
- 2) *If e is a section of $\mathcal{O}_X(E)$ defining E , *i.e.*, $E = (e = 0)$, then the cohomology class ξ is annihilated by e , *i.e.*,*

$$e\xi = 0 \quad \text{in} \quad H^1(\Theta_X(E - K_X - M)).$$

- 3) *If, in addition, $X \subset \mathbb{P}^4$ lies on a 3-fold of degree $m \leq 5$ and $\xi = \delta_X(s)$, where δ_X is the coboundary map in (2.3), then*

$$H^0(\Theta_{\mathbb{P}^4}|_X(E - K_X - (5 - m)H)) = H^0(\Theta_{\mathbb{P}^4}|_X(-L - (5 - m)H)) \neq 0.$$

In particular, $H \cdot L \leq (m - 4)H^2$.

Proof. Set

$$\varphi_\xi : \mathcal{F} \longrightarrow \Omega_X$$

to be the morphism defined by the composition of the inclusion $\mathcal{F} \hookrightarrow \mathcal{T}_\xi$ together with the epimorphism of the extension sequence (2.7). By assumption φ_ξ is generically an isomorphism.

This implies that the support of $\text{coker}(\varphi_\xi)$ is a nonzero effective divisor, since otherwise φ_ξ is an isomorphism and the exact sequence (2.7) splits or, equivalently, $\xi = 0$.

Writing out the exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi_\xi} \Omega_X \longrightarrow \text{coker}(\varphi_\xi) \longrightarrow 0$$

we obtain the decomposition of the canonical divisor asserted in 1) of the lemma.

To prove 2) we consider the diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathcal{F} & & & \\
 & & & \downarrow & \searrow \varphi_\xi & & \\
 0 \longrightarrow \mathcal{O}_X(-K_X - M) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & \searrow e & \downarrow & & & & \\
 & & \mathcal{J}_Z(-L - M) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array} \tag{2.8}$$

where the slanted arrow in the lower part of the diagram is the morphism given by multiplication with the section e . Dualizing and tensoring the diagram with $\mathcal{O}_X(-L - M)$ we arrive at

$$\begin{array}{ccccccc}
 & & & \mathcal{O}_X & & & \\
 & & & \downarrow & \searrow e & & \\
 0 \longrightarrow \Theta_X(-L - M) & \longrightarrow & \mathcal{T}_\xi^*(-L - M) & \longrightarrow & \mathcal{O}_X(K_X - L) & \longrightarrow & 0
 \end{array} \tag{2.9}$$

Since the coboundary map

$$H^0(\mathcal{O}_X(K_X - L)) = H^0(\mathcal{O}_X(E)) \longrightarrow H^1(\Theta_X(-L - M))$$

in long exact sequence of cohomology groups of the horizontal sequence in (2.9) is given by the cup-product with the class ξ and since the section e is in its kernel, we deduce $e\xi = 0$ in $H^1(\Theta_X(-L - M))$.

For 3), we use the fact that $\xi = \delta_X(s)$, where δ_X is as in (2.3). Set $\Delta = K_X + (5 - m)H$ and consider the commutative diagram

$$\begin{array}{ccccc}
 H^0(\mathcal{N}_X(-\Delta)) & \xrightarrow{\delta_X} & H^1(\Theta_X(-\Delta)) & & \\
 e \downarrow & & \downarrow e & & \\
 H^0(\Theta_{\mathbb{P}^4|X}(E - \Delta)) & \longrightarrow & H^0(\mathcal{N}_X(E - \Delta)) & \xrightarrow{\delta'_X} & H^1(\Theta_X(E - \Delta))
 \end{array}$$

From this and 2) of the lemma, we obtain

$$\delta'_X(es) = e\delta_X(s) = e\xi = 0.$$

Hence the global section $es \in H^0(\mathcal{N}_X(E - K_X - (5 - m)H))$, being obviously nonzero, comes from a nonzero section in $H^0(\Theta_{\mathbb{P}^4|X}(E - K_X - (5 - m)H))$. This proves the first assertion of 3).

To see the inequality $H \cdot L \leq (m - 4)H^2$, we restrict the Euler sequence for \mathbb{P}^4 to X and tensor it with $\mathcal{O}_X(E - K_X - (5 - m)H) = \mathcal{O}_X(-L - (5 - m)H)$ to arrive at

$$0 \rightarrow \mathcal{O}_X(-L - (5 - m)H) \rightarrow H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X(-L - (4 - m)H) \rightarrow \Theta_{\mathbb{P}^4|X}(-L - (5 - m)H) \rightarrow 0.$$

Since $H^0(\Theta_{\mathbb{P}^4|X}(E - K_X - (5 - m)H)) \neq 0$, by the first part of 3), the above sequence implies that either $H^0(\mathcal{O}_X(-L - (4 - m)H)) \neq 0$, or

$$\ker \left(H^1(\mathcal{O}_X(-L - (5 - m)H)) \rightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(-L - (4 - m)H)) \right) \neq 0.$$

The first possibility immediately gives the assertion $H \cdot L \leq (m - 4)H^2$. The second one implies that the homomorphism $H^1(\mathcal{O}_X(-L - (5 - m)H)) \xrightarrow{h} H^1(\mathcal{O}_X(-L - (4 - m)H))$ given by the multiplication by any global section $h \in H^0(\mathcal{O}_X(H))$ has a nonzero kernel. Since that kernel comes from $H^0(\mathcal{O}_{C_h}(-L - (4 - m)H))$, where $C_h = (h = 0)$, we deduce that $H^0(\mathcal{O}_C(-L - (4 - m)H)) \neq 0$, for any divisor C in the linear system $|H|$. This implies $H \cdot (-L - (4 - m)H) \geq 0$ and hence the assertion $H \cdot L \leq (m - 4)H^2$. \square

(I) Hartshorne-Lichtenbaum for surfaces of general type

3. NUMERICAL INVARIANTS FOR SURFACES ON DEGREE 4 HYPERSURFACES

In this section we prove the inequalities (1.5) of the form $c_2 - K_X^2 \geq aH \cdot K_X + bd$, stated in the introduction. Let X be a smooth surface in \mathbb{P}^4 lying on a 3-fold $V = V_4$ of degree four and not on any of a smaller degree. Unless stated otherwise, we assume that the Kodaira dimension of X is non-negative. This assumption, according to Lemma 2.2, gives a nonzero cohomology class $\delta_X(s) \in H^1(\Theta_X(-K_X - H))$ (see Lemma 2.2 for notation) which we denote by ξ . As we already explained, this cohomology class is used to build the extension (2.7) and the focus of study becomes the vector bundle \mathcal{T}_ξ sitting in the middle of that sequence. The inequalities we are after are a consequence of Bogomolov semistability or instability of \mathcal{T}_ξ .

3.1. The inequalities (1.5)

THEOREM 3.1. *Let \mathcal{T}_ξ be the sheaf in the middle of the extension sequence (2.7) associated to $\xi = \delta_X(s)$.*

- 1) *If \mathcal{T}_ξ is Bogomolov semistable, then $c_2 - K_X^2 \geq H \cdot K_X + \frac{1}{3}d$.*
- 2) *If \mathcal{T}_ξ is Bogomolov unstable, then $c_2 - K_X^2 \geq \min \left(\frac{3}{4}H \cdot K_X, \frac{1}{2}H \cdot K_X + \frac{1}{4}d \right)$.*

Proof. From the exact sequence (2.7) it follows that \mathcal{T}_ξ is a locally free sheaf of rank 3 with the Chern invariants $c_1(\mathcal{T}_\xi) = -H$ and $c_2(\mathcal{T}_\xi) = c_2 - K_X^2 - H \cdot K_X$. Therefore the Bogomolov semistability condition for \mathcal{T}_ξ reads as follows

$$6c_2(\mathcal{T}_\xi) - 2c_1^2(\mathcal{T}_\xi) = 6(c_2 - K_X^2) - 6H \cdot K_X - 2H^2 \geq 0$$

and this is equivalent to the inequality in 1) of the theorem.

We now turn to the case when \mathcal{T}_ξ is Bogomolov unstable. To analyse the situation we use the Bogomolov filtration of \mathcal{T}_ξ , see Lemma 2.1. In particular, according to the shape of that filtration, we obtain the following inequalities⁶:

$$c_2 - K_X^2 \geq \begin{cases} H \cdot K_X, & \text{if } 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{T}_\xi \text{ and } \text{rank}(\mathcal{F}_1) = 2 \\ \frac{1}{2} H \cdot K_X + \frac{1}{4} d, & \text{if } 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{T}_\xi \text{ and } \text{rank}(\mathcal{F}_1) = 1 \\ \frac{3}{4} H \cdot K_X, & \text{if } 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathcal{T}_\xi. \end{cases} \quad (3.1)$$

This implies 2) of the theorem. \square

Before we proceed with the proof of (3.1), we would like to provide the reader with the conducting line of the proofs of the lemmas below. The basic idea is to use the Bogomolov filtration of \mathcal{T}_ξ for writing down a “good” lower bound for the second Chern number of \mathcal{T}_ξ . “Good” here means that a sought after estimate should imply a lower bound for $c_2 - K_X^2$ as a positive function of d or/and $H \cdot K_X$. This is possible in view of the following special features of \mathcal{T}_ξ :

- The subsheaves of rank 2 involved in the filtration of \mathcal{T}_ξ satisfy the hypotheses of the technical Lemma 2.3.
- The subsheaves of rank 1 involved in the filtration of \mathcal{T}_ξ having positive degree (with respect to some polarization of X) inject into Ω_X and hence must be of Iitaka dimension at most one (Bogomolov lemma); furthermore, the generic semi-positivity of Ω_X insures that the quotient sheaf must be of non-negative degree.

With these remarks in mind, we now consider all possible filtrations of the Bogomolov unstable vector bundle \mathcal{T}_ξ .

LEMMA 3.2. *If the Bogomolov filtration of \mathcal{T}_ξ is $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{T}_\xi$ with \mathcal{F}_1 of rank 2, then it gives rise to a divisor B_1 in the positive cone of $N(X)$ and to an effective nonzero divisor E such that the following hold:*

- 1) $c_1(\mathcal{F}_1) = \frac{1}{3} (B_1 - 2H)$ and $B_1 \cdot H \leq 2d$,
- 2) $K_X = \frac{1}{3} (B_1 - 2H) + E$,
- 3) $c_2 - K_X^2 \geq H \cdot K_X$.

Proof. The bundle \mathcal{T}_ξ is the middle term of two exact sequences, as in diagram (2.8), where the maximal Bogomolov destabilizing subsheaf \mathcal{F}_1 takes the place of \mathcal{F} .

We set $L_1 = c_1(\mathcal{F}_1)$ and $L_2 = c_1(\mathcal{T}_\xi/\mathcal{F}_1)$. Using the equality

$$-H = c_1(\mathcal{T}_\xi) = L_1 + L_2,$$

the Bogomolov destabilizing condition for \mathcal{F}_1 tells us that the \mathbb{Q} -divisor

$$\frac{c_1(\mathcal{F})}{2} - \frac{c_1(\mathcal{T}_\xi)}{3} = \frac{L_1}{2} - \frac{-H}{3} = \frac{1}{6} (L_1 - 2L_2),$$

⁶The inequality of the last line in (3.1) is strict, see Lemma 3.4 for details.

lies in the positive cone $N^+(X)$ of $N(X)$. Thus the divisor $B_1 = L_1 - 2L_2$ lies in the positive cone $N^+(X)$ and we write L_i , $i = 1, 2$, as a linear combination of H and B_1 as follows:

$$\begin{aligned} L_1 &= -\frac{2}{3}H + \frac{1}{3}B_1 \\ L_2 &= -\frac{1}{3}H - \frac{1}{3}B_1. \end{aligned} \tag{3.2}$$

Recall that $c_2(\mathcal{T}_\xi) = c_2 - K_X^2 - H \cdot K_X$. Computing that Chern class using the vertical sequence in (2.8) with $\mathcal{F} = \mathcal{F}_1$ and the quotient sheaf $\mathcal{T}_\xi/\mathcal{F}_1 = \mathcal{I}_Z(L_2)$, we obtain

$$c_2 - K_X^2 - H \cdot K_X = c_2(\mathcal{F}_1) + L_1 \cdot L_2 + \deg(Z) \geq \frac{1}{4}L_1^2 + L_1 \cdot L_2 = \frac{1}{12}(4H^2 - B_1^2). \tag{3.3}$$

where the inequality uses $c_2(\mathcal{F}_1) \geq \frac{1}{4}L_1^2$, the Bogomolov semistability of \mathcal{F}_1 , and the last equality comes from substituting the expressions from (3.2).

Next we claim that the slanted arrow $\varphi_\xi : \mathcal{F}_1 \rightarrow \Omega_X$ in the diagram (2.8) is generically an isomorphism. Indeed, if φ_ξ drops its rank everywhere, then we obtain the commutative diagram

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ 0 & \longrightarrow & \mathcal{O}_X(-K_X - H) & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \text{im}(\varphi_\xi) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X(-K_X - H) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X \longrightarrow 0 \end{array}$$

where the sheaf $\text{im}(\varphi_\xi)$ is the image of φ_ξ . It is a torsion free subsheaf of rank 1 of Ω_X with the first Chern class

$$c_1(\text{im}(\varphi_\xi)) = c_1(\mathcal{F}_1) + (K_X + H) = -\frac{2}{3}H + \frac{1}{3}B_1 + K_X + H = K_X + \frac{1}{3}H + \frac{1}{3}B_1.$$

But this means that Ω_X contains a rank 1 subsheaf of Iitaka dimension 2 which is impossible in view of Bogomolov Lemma.

Once we know that φ_ξ is generically of maximal rank, Lemma 2.3 can be applied. In particular, we obtain the decomposition asserted in 2) of that lemma, where the divisor E is the support of the cokernel of φ_ξ , and, from the part 3) of Lemma 2.3, we deduce that $H \cdot L_1 \leq 0$. This together with the formula for L_1 in (3.2) implies

$$0 \geq H \cdot L_1 = H \cdot \left(-\frac{2}{3}H + \frac{1}{3}B_1\right) = \frac{1}{3}(B_1 \cdot H - 2d).$$

Hence $B_1 \cdot H \leq 2d$ as asserted in 1) of the lemma.

The above inequality and the Hodge Index Theorem give $B_1^2 \leq 4d$. Substituting this inequality in (3.3), we deduce assertion 3) of the lemma. \square

LEMMA 3.3. *If the Bogomolov filtration of \mathcal{T}_ξ is $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{T}_\xi$ with \mathcal{F}_1 of rank 1, then*

$$c_2 - K_X^2 \geq \begin{cases} H \cdot K_X + \frac{1}{4}d, & \text{if } H \cdot c_1(\mathcal{F}_1) \leq 0 \\ \frac{1}{2}H \cdot K_X + \frac{1}{4}d, & \text{if } H \cdot c_1(\mathcal{F}_1) > 0. \end{cases}$$

Proof. Let $\mathcal{Q} = \mathcal{T}_\xi/\mathcal{F}_1$ and set $L_1 = c_1(\mathcal{F}_1)$ and $L_2 = c_1(\mathcal{Q})$. The bundle \mathcal{T}_ξ becomes the middle term of two exact sequences.

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{O}_X(L_1) & & & & \\
 & & \downarrow & \searrow & & & \\
 0 & \longrightarrow & \mathcal{O}_X(-K_X - H) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X \longrightarrow 0 \\
 & & \searrow & & \downarrow & & \\
 & & & & \mathcal{Q} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array} \tag{3.4}$$

The condition that $\mathcal{O}_X(L_1)$ is a Bogomolov destabilizing subsheaf of \mathcal{T}_ξ and the equality $(-H) = L_1 + L_2$ imply that the \mathbb{Q} -divisor

$$L_1 - \frac{c_1(\mathcal{T}_\xi)}{3} = L_1 - \frac{-H}{3} = L_1 - \frac{L_1 + L_2}{3} = \frac{2}{3} \left(L_1 - \frac{1}{2} L_2 \right)$$

lies in the positive cone of $N(X)$. Setting

$$B_1 = L_1 - \frac{1}{2} L_2 \in N^+(X),$$

we express L_1 and L_2 as linear combinations of H and B_1 :

$$\begin{aligned}
 L_1 &= -\frac{1}{3} H + \frac{2}{3} B_1 \\
 L_2 &= -\frac{2}{3} H - \frac{2}{3} B_1.
 \end{aligned} \tag{3.5}$$

From here on we use the same argument as in Lemma 3.2. Namely, we use the vertical sequence in (3.4) to estimate the second Chern class of \mathcal{T}_ξ :

$$c_2 - K_X^2 - H \cdot K_X = c_2(\mathcal{Q}) + L_1 \cdot L_2 \geq \frac{1}{4} L_2^2 + L_1 \cdot L_2 = \frac{1}{3} (d - B_1^2), \tag{3.6}$$

where the inequality uses the condition that the quotient $\mathcal{Q} = \mathcal{F}_2/\mathcal{F}_1$ is Bogomolov semi-stable, *i.e.*, $c_2(\mathcal{Q}) \geq \frac{1}{4} L_2^2$, and the last equality comes from substituting the expressions from (3.5).

To conclude the argument we need an appropriate upper bound on the self-intersection B_1^2 . We argue according to the sign of $H \cdot L_1$.

First case. If $H \cdot L_1 \leq 0$, then we are essentially in the same situation as in the proof of Lemma 3.2. Namely, we have

$$0 \geq H \cdot L_1 = \frac{1}{3} H \cdot (-H + 2B_1) = \frac{1}{3} (2B_1 \cdot H - d),$$

where the first equality uses the formula for L_1 in (3.5). Thus we obtain $H \cdot B_1 \leq \frac{1}{2} d$ and hence, by the Hodge index, the upper bound $B_1^2 \leq \frac{1}{4} d$. The inequality (3.6) then becomes

$$c_2 - K_X^2 - H \cdot K_X \geq \frac{1}{3} (d - B_1^2) \geq \frac{1}{3} \left(d - \frac{1}{4} d \right) = \frac{1}{4} d$$

and this is equivalent to the first inequality of the lemma.

Second case. If $H \cdot L_1 > 0$, then the morphism $\mathcal{O}_X(L_1) \rightarrow \Omega_X$ given by slanted arrow in the diagram (3.4) is nonzero.⁷ Hence $\mathcal{O}_X(L_1)$ injects into Ω_X and by the Bogomolov Lemma, $L_1^2 \leq 0$. This inequality and the formula (3.5) for L_1 give

$$0 \geq (-H + 2B_1)^2 = H^2 - 4H \cdot B_1 + 4B_1^2,$$

or equivalently,

$$B_1^2 \leq H \cdot B_1 - \frac{1}{4}d. \quad (3.7)$$

On the other hand, the generic semi-positivity of Ω_X stipulates that the quotient sheaf $\Omega_X/\mathcal{O}_X(L_1)$ has non-negative degree with respect to any ample divisor on X . In particular, we deduce that

$$0 \leq H \cdot c_1(\Omega_X/\mathcal{O}_X(L_1)) = H \cdot K_X - H \cdot L_1. \quad (3.8)$$

The above inequality and the formula for L_1 in (3.5) imply

$$H \cdot B_1 \leq \frac{1}{2}(3H \cdot K_X + d). \quad (3.9)$$

Combining (3.7) and (3.9) we obtain

$$B_1^2 \leq \frac{3}{2}H \cdot K_X + \frac{1}{4}d.$$

This inequality together with (3.6) gives the estimate

$$c_2 - K_X^2 - H \cdot K_X \geq -\frac{1}{2}H \cdot K_X + \frac{1}{4}d,$$

which is equivalent to the second inequality of the lemma. \square

LEMMA 3.4. *If the Bogomolov filtration of \mathcal{T}_ξ is $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathcal{T}_\xi$, then it determines two divisors B_i , $i = 1, 2$, in the positive cone of $N(X)$ and an effective nonzero divisor E such that the following conditions hold:*

- 1) $c_1(\mathcal{F}_1) = \frac{1}{3}(2B_1 + B_2 - H)$,
- 2) $K_X = \frac{1}{3}(B_1 + 2B_2 - 2H) + E$,
- 3) $c_2 - K_X^2 \geq \begin{cases} H \cdot K_X + B_1^2 + B_1 \cdot B_2, & \text{if } H \cdot c_1(\mathcal{F}_1) \leq 0 \\ \frac{3}{4}H \cdot K_X + \frac{1}{8} \frac{(H \cdot B_1)(H \cdot B_2)}{d} & \text{if } H \cdot c_1(\mathcal{F}_1) > 0. \end{cases}$

Proof. Set $L_i = c_1(\mathcal{F}_i/\mathcal{F}_{i-1})$, $1 \leq i \leq 3$. Hence $c_1(\mathcal{F}_i) = L_1 + \dots + L_i$, for $i = 1, 2, 3$. In particular,

$$-H = c_1(\mathcal{T}_\xi) = c_1(\mathcal{F}_3) = L_1 + L_2 + L_3.$$

Since \mathcal{F}_1 is the maximal destabilizing subsheaf of \mathcal{F}_2

$$B_1 = L_1 - L_2 = 2 \left(L_1 - \frac{c_1(\mathcal{F}_2)}{2} \right) \in N^+(X), \quad (3.10)$$

⁷Otherwise the divisor $-(H + K_X + L_1)$ must be effective; however substituting the formula for L_1 from (3.5), one obtains $H + K_X + L_1 = \frac{3}{4}H + K_X + \frac{2}{3}B_1$ which is of the Iitaka dimension 2.

and similarly, since $\mathcal{F}_2/\mathcal{F}_1$ is the maximal destabilizing subsheaf of $\mathcal{F}_3/\mathcal{F}_1$,

$$B_2 = L_2 - L_3 = 2 \left(c_1(\mathcal{F}_2/\mathcal{F}_1) - \frac{c_1(\mathcal{F}_3/\mathcal{F}_1)}{2} \right) \in N^+(X). \quad (3.11)$$

Combining (3.11) and (3.10) together with the decomposition $H = -L_1 - L_2 - L_3$, we obtain the following formulas

$$\begin{aligned} L_1 &= -\frac{1}{3}H + \frac{2}{3}B_1 + \frac{1}{3}B_2 \\ L_2 &= -\frac{1}{3}H - \frac{1}{3}B_1 + \frac{1}{3}B_2 \\ L_3 &= -\frac{1}{3}H - \frac{1}{3}B_1 - \frac{2}{3}B_2. \end{aligned} \quad (3.12)$$

As in the two previous lemmas, we use the Bogomolov filtration of \mathcal{T}_ξ to estimate its second Chern number $c_2(\mathcal{T}_\xi) = c_2 - K_X^2 - H \cdot K_X$. Since the filtration is a maximal ladder, it yields

$$c_2 - K_X^2 - H \cdot K_X \geq L_1 \cdot L_2 + L_1 \cdot L_3 + L_2 \cdot L_3.$$

Substituting the formulas from (3.12) leads to

$$c_2 - K_X^2 - H \cdot K_X \geq \frac{1}{3}(d - B_1^2 - B_2^2 - B_1 \cdot B_2) = \frac{1}{3} \left(d - \frac{1}{4}(2B_1 + B_2)^2 - \frac{3}{4}B_2^2 \right). \quad (3.13)$$

The argument continues, as in the proof of Lemma 3.3, according to the sign of the intersection $H \cdot L_1$.

First case. If $H \cdot L_1 \leq 0$, then the formula for L_1 in (3.12) gives $d \geq 2H \cdot B_1 + H \cdot B_2 = H \cdot (2B_1 + B_2)$. This and the Hodge Index Theorem imply

$$(2B_1 + B_2)^2 \leq d. \quad (3.14)$$

As a consequence, the inequality (3.13) becomes

$$c_2 - K_X^2 - H \cdot K_X \geq \frac{1}{4}(d - B_2^2) \geq B_1^2 + B_1 \cdot B_2,$$

where the last inequality is obtained by substituting the upper bound for B_2^2 from (3.14). Hence the first inequality in part 3) of the lemma.

Second case. If $H \cdot L_1 > 0$, then as in the proof of Lemma 3.3, we obtain $L_1^2 \leq 0$. This and the formula for L_1 in (3.12) imply

$$2H \cdot (2B_1 + B_2) \geq d + (2B_1 + B_2)^2. \quad (3.15)$$

Next we exploit the subsheaf \mathcal{F}_2 of the Bogomolov filtration of \mathcal{T}_ξ . Namely, combining the inclusion $\mathcal{F}_2 \subset \mathcal{T}_\xi$ with the epimorphism of the extension sequence (2.7) gives rise to a nonzero morphism

$$u : \mathcal{F}_2 \longrightarrow \Omega_X.$$

Arguing exactly as in the proof of Lemma 3.2 one shows that this morphism is generically an isomorphism. Thus we can apply Lemma 2.3 to obtain the decomposition

$$K_X = c_1(\mathcal{F}_2) + E = L_1 + L_2 + E = -H - L_3 + E,$$

where E is the support of the cokernel of u . This is the decomposition asserted in part 2) of the lemma.

Furthermore, the last assertion in Lemma 2.3 tells us that $(-H - L_3) \cdot H \leq 0$. This inequality combined with the formula for L_3 in (3.12) gives the following inequality:

$$0 \leq H \cdot (H + L_3) = d - \frac{1}{3} H \cdot (H + B_1 + 2B_2) = \frac{1}{3} (2d - H \cdot B_1 - 2H \cdot B_2).$$

Thus

$$H \cdot B_1 + 2H \cdot B_2 \leq 2d. \quad (3.16)$$

Rewriting this inequality as $H \cdot B_2 \leq d - \frac{1}{2} H \cdot B_1$, implies that $H \cdot B_2 < d$ and hence, by the Hodge Index Theorem, we obtain

$$B_2^2 \leq \frac{H \cdot B_2}{H^2} H \cdot B_2 \leq \frac{H \cdot B_2}{d} \left(d - \frac{1}{2} H \cdot B_1 \right) = H \cdot B_2 - \frac{1}{2d} (H \cdot B_1)(H \cdot B_2). \quad (3.17)$$

We now return to the inequality (3.13). Substituting the upper bounds for the second (resp. third) term from (3.15) (resp. (3.17)), we obtain

$$\begin{aligned} c_2 - K_X^2 - H \cdot K_X &\geq \frac{1}{12} (5d - 4H \cdot B_1 - 5H \cdot B_2) + \frac{1}{8d} (H \cdot B_1)(H \cdot B_2) \\ &= \frac{1}{12} (4d - 2H \cdot B_1 - 4H \cdot B_2) + \frac{1}{12} (d - 2H \cdot B_1 - H \cdot B_2) + \frac{1}{8d} (H \cdot B_1)(H \cdot B_2) \\ &\geq -\frac{1}{4} H \cdot L_1 + \frac{1}{8d} (H \cdot B_1)(H \cdot B_2), \end{aligned}$$

where the last inequality is obtained by using (3.16) for the first parenthesis, and the formula for L_1 in (3.12) for the second one. Thus we obtain

$$c_2 - K_X^2 \geq \frac{3}{4} H \cdot K_X + \frac{1}{4} (K_X - L_1) \cdot H + \frac{1}{8d} (H \cdot B_1)(H \cdot B_2) \geq \frac{3}{4} H \cdot K_X + \frac{1}{8d} (H \cdot B_1)(H \cdot B_2),$$

where the last inequality uses $(K_X - L_1) \cdot H \geq 0$ coming from the generic semi-positivity of Ω_X , see the discussion just above (3.8). This completes the proof of the second inequality in part 3) of the lemma. \square

As a corollary of Theorem 3.1 we deduce the following property of surfaces lying on a quartic hypersurface in \mathbb{P}^4 .

THEOREM 3.5. *If $X \subset \mathbb{P}^4$ is a smooth surface lying on a hypersurface of degree four and not on one of a smaller degree, then $K_X^2 < 6\chi(\mathcal{O}_X)$.*

Proof. Assume $K_X^2 \geq 6\chi(\mathcal{O}_X)$. We claim that X must be rational or rationally ruled. Indeed, if this is not the case, Theorem 3.1 tells us that $K_X^2 = 6\chi(\mathcal{O}_X)$ and $H \cdot K_X = 0$. The second identity implies that $K_X = 0$ in $N(X)$. Hence $K_X^2 = \chi(\mathcal{O}_X) = 0$. In particular, X must be irregular with irregularity $q = 1$ or 2 , and of degree $d = 10$ (this is obtained from the double point formula).

Furthermore, Theorem 3.1 tells us that the vector bundle \mathcal{T}_ξ is Bogomolov unstable with the filtration treated in Lemma 3.2. From the proof of that lemma we deduce the isomorphisms

$$H^0(\Omega_X) \cong H^0(\mathcal{T}_\xi) \cong H^0(\mathcal{F}_1).$$

This implies that we must have $q = 1$ (otherwise X is an abelian surface and the above isomorphisms imply $\mathcal{F}_1 \cong \Omega_X$, hence a splitting of the extension sequence (2.7)). Thus the Albanese variety $\text{Alb}(X)$ is an elliptic curve, call it A , and the Albanese map $a : X \rightarrow A$ is an elliptic fibration. In addition, since $c_2(X) = 0$, we also see that every *reduced* fibre is smooth. In particular, there are no smooth rational curves on X . This fact, in turn, implies that any divisor in the positive cone of $N(X)$ is ample.

Another aspect of the filtration in Lemma 3.2 is the decomposition of the canonical divisor

$$0 = K_X = L_1 + E = \frac{1}{3}(B_1 - 2H) + E,$$

where $L_1 = c_1(\mathcal{F}_1) = \frac{1}{3}(B_1 - 2H)$, see (3.1), and E is an effective nonzero divisor. The above equation can be rewritten as follows

$$2H = 3E + B_1, \tag{3.18}$$

with B_1 in the positive cone. Hence, as remarked above, the divisor B_1 is ample.

Next we investigate the divisor E . Using formula (3.18), we deduce that

$$C \cdot E + \frac{1}{3}C \cdot B_1 = \frac{2}{3}H \cdot C,$$

for any reduced irreducible component C of E . On the other hand, we know that for every C as above the subsheaf $\mathcal{F}_1 \subset \mathcal{T}_\xi$ gives rise to a nonzero morphism (see [26])

$$\mathcal{O}_C(L_1 + H) \longrightarrow \Theta_X \otimes \mathcal{O}_C. \tag{3.19}$$

Using again the expression for L_1 from (3.2), we obtain

$$(L_1 + H) \cdot C = \frac{1}{3}(H + B_1) \cdot C > 0. \tag{3.20}$$

Hence the composition of the morphism (3.19) with the morphism $\Theta_X \otimes \mathcal{O}_C \rightarrow \mathcal{O}_C(C)$ coming from the normal sequence of $C \subset X$ yields a nonzero morphism⁸

$$\mathcal{O}_C(L_1 + H) \longrightarrow \mathcal{O}_C(C).$$

This morphism together with formula (3.20) leads to

$$C^2 \geq (L_1 + H) \cdot C = \frac{1}{3}(H + B_1) \cdot C > 0, \tag{3.21}$$

implying that E lies in the positive cone of $N(X)$ and hence it is ample.

From (3.18), it follows that

$$20 = 2H^2 = 3E \cdot H + B_1 \cdot H \geq 3E \cdot H + 4.$$

Thus $E \cdot H \leq 5$ and by the Hodge Index Theorem, we obtain $E^2 \leq 2$. Since the intersection form on $\text{NS}(X)$ is even and E is ample, we deduce the equality $E^2 = 2$. Hence E is reduced and irreducible. Thus replacing C by E in (3.21), we obtain

$$6 = 3E^2 \geq H \cdot E + B_1 \cdot E.$$

⁸The fact that X contains no smooth rational curve is used here again.

But the Hodge index tells us that $H \cdot E = 5$ and that $B_1 \cdot E \geq 2$ which contradicts the above inequality.

Now we know that X is either rational or rationally ruled. In the latter case $\chi(\mathcal{O}_X) \leq 0$ and $8\chi(\mathcal{O}_X) \geq K_X^2 \geq 6\chi(\mathcal{O}_X)$, implying $K_X^2 = \chi(\mathcal{O}_X) = 0$. This identifies X as the projectivization $\mathbb{P}(\mathcal{E})$ of a rank 2 bundle \mathcal{E} over an elliptic curve. But then it is well-known that X is an elliptic scroll of degree $d = -H \cdot K_X = 5$, see Lemma 7.8. It is easy to see (and again well-known, see Theorem 8.1) that such an X is contained in cubic hypersurfaces. This of course is contrary to our assumption that 4 is the smallest degree of a hypersurface containing X .

We now turn to the remaining possibility: X is rational. In this case $\chi(\mathcal{O}_X) = 1$ and hence, $K_X^2 \geq 6\chi(\mathcal{O}_X) = 6$. By Riemann-Roch applied to $\mathcal{O}_X(-K_X)$ it follows that $h^0(\mathcal{O}_X(-K_X)) \geq 7$. Therefore, $(-K_X)$ is an effective nonzero divisor.

Next we wish to have an upper bound for $(-H \cdot K_X)$. This can be done by observing that

$$h^0(\mathcal{O}_X(H)) \geq \chi(\mathcal{O}_X(H)) = \frac{H \cdot (H - K_X)}{2} + 1,$$

which yields

$$d - H \cdot K_X \leq 2h^0(\mathcal{O}_X(H)) - 2.$$

Furthermore, it is well-known that $h^0(\mathcal{O}_X(H)) = 5$, unless X is the projection of the Veronese surface from a general point in \mathbb{P}^5 . But such a projection lies on a hypersurface of degree 3 and this is contrary to our assumption that the smallest degree of a hypersurface containing X is 4. Thus $h^0(\mathcal{O}_X(H)) = 5$ and we obtain the upper bound

$$-H \cdot K_X \leq 8 - d. \quad (3.22)$$

However, one observes the following.

Claim. $H^0(\mathcal{O}_X(-K_X - H)) \neq 0$.

This concludes the argument, since $H^0(\mathcal{O}_X(-K_X - H)) \neq 0$ together with the earlier estimate $h^0(\mathcal{O}_X(-K_X)) \geq 7$ imply $(-H \cdot K_X) > d$. From this inequality and (3.22), one obtains $d = 3$. It follows immediately that X is the projection of the Veronese surface from a point on it. But then, an easy dimension count implies that X lies on a hypersurface of degree 2, contradicting our hypothesis.

We now return to the proof of the claim. It is equivalent to showing that the homomorphism $H^0(\mathcal{O}_X(-K_X)) \rightarrow H^0(\mathcal{O}_C(-K_X))$ induced by the restriction of sections to a smooth irreducible curve $C \in |H|$ is not injective. Assume on the contrary that it is injective. Then

$$7 \leq h^0(\mathcal{O}_X(-K_X)) \leq h^0(\mathcal{O}_C(-K_X)).$$

If $\mathcal{O}_C(-K_X)$ is non-special, then we compute the right hand side in the above inequality from the Riemann-Roch and obtain

$$7 \leq h^0(\mathcal{O}_X(-K_X)) \leq h^0(\mathcal{O}_C(-K_X)) = \deg(-K_X|_C) + 1 - g(C) = \frac{-3H \cdot K_X - d}{2}.$$

Equivalently, $-H \cdot K_X \geq \frac{14 + d}{3}$, which together with (3.22) tells us that $d \leq 2$ which is impossible since d must be at least 3.

If $\mathcal{O}_C(-K_X)$ is special, then by the Clifford inequality

$$14 \leq 2h^0(\mathcal{O}_X(-K_X)) \leq 2h^0(\mathcal{O}_C(-K_X)) \leq -H \cdot K_X + 2.$$

Hence $(-H \cdot K_X) \geq 12$, which again is impossible in view of the upper bound (3.22). \square

3.2. The proof of Theorem 1.3 when $m_X = 4$

For the reader's convenience we restate the theorem in this case as follows.

PROPOSITION 3.6. *For all surfaces of general type contained in a quartic hypersurface in \mathbb{P}^4 and not in one of a smaller degree the following assertions hold.*

- 1) *The slope of their Chern numbers $\alpha = \frac{K^2}{\chi} < 6$.*
- 2) *For every rational number $\alpha < 6$, there exists $d(\alpha) \in \mathbb{N}$ such that every surface of slope α has the degree $d \leq d(\alpha)$.*
- 3) *For every rational number $\alpha < 6$, all surfaces of the slope α and degree d have the holomorphic Euler characteristic $\chi \leq \chi(\alpha, d)$, where*

$$\chi(\alpha, d) = \frac{d^2 + 20d}{8(6 - \alpha)}.$$

Proof. Let X be a smooth surface in \mathbb{P}^4 with $m_X = 4$, see (1.3) for notation. The first assertion of the proposition, $\alpha_X < 6$, is a reformulation of Theorem 3.5.

For the second assertion, we use the inequality (1.4) for $m = m_X = 4$ to obtain

$$\chi := \chi(\mathcal{O}_X) \geq \frac{1}{96} d^3 - \frac{19}{16} d^2 + \frac{10}{3} d + \frac{5}{4}, \quad (3.23)$$

provided $d \geq 11$.

On the other hand the Chern numbers of X are related to the degree d via the double point formula which we write in the form

$$12\chi - 2K_X^2 = c_2 - K_X^2 = 5H \cdot K_X + 10d - d^2.$$

Setting $K_X^2 = \alpha\chi$ and substituting into the above equation give

$$5H \cdot K_X + 10d - d^2 = 12\chi - 2K_X^2 = 12\chi - 2\alpha\chi = 2(6 - \alpha)\chi. \quad (3.24)$$

From [15] the sectional genus of X has the following upper bound

$$2g(H) - 2 = H^2 + H \cdot K_X \leq \frac{1}{4} d^2. \quad (3.25)$$

Using this in the equation (3.24) we obtain

$$2(6 - \alpha)\chi = 5H \cdot K_X + 10d - d^2 = 5(H \cdot K_X + d) + 5d - d^2 \leq \frac{1}{4} d^2 + 5d. \quad (3.26)$$

This inequality combined with the first assertion of the proposition and the lower bound for χ in (3.23) give

$$(6 - \alpha) \left(\frac{1}{12} d^3 - \frac{19}{2} d^2 + \frac{80}{3} d + 10 \right) \leq d^2 + 20d, \quad (3.27)$$

for all $d \geq 11$. From this it follows that one can explicitly determine a positive integer $d_0(\alpha)$ depending on α only, which is an upper bound for the solutions of the above inequality. Setting $d(\alpha) = \max(10, d_0(\alpha))$, one deduces the second assertion of the proposition.

The upper bound for χ in the third assertion of the proposition is obtained from (3.26) and the first assertion of the proposition. \square

As a consequence, we deduce the following finiteness result.

COROLLARY 3.7. *The number of families of surfaces in \mathbb{P}^4 of general type with fixed slope α and contained in a quartic hypersurface (and not in one of a smaller degree) is at most finite.*

Proof. The components of the Hilbert scheme of surfaces in \mathbb{P}^4 are labeled by Hilbert polynomials. So the proof of the statement comes down to checking that there is at most a finite number of such polynomials for the surfaces subject to the hypotheses of the corollary. Since the Hilbert polynomial of a surface $X \subset \mathbb{P}^4$ of degree d has the form

$$P_X(t) = \frac{d}{2}t^2 - \frac{H \cdot K_X}{2}t + \chi(\mathcal{O}_X),$$

one needs to see that there is only a finite number of possibilities for the triples $(d, H \cdot K_X, \chi(\mathcal{O}_X))$. This is exactly what Proposition 3.6 tells us: parts 2) and 3) give a finite number of possibilities for d and $\chi(\mathcal{O}_X)$ respectively. Once d takes a finite number of values, the inequality (3.25) insures that there is only a finite number of values for $H \cdot K_X$ as well. \square

Remarks. 1) If $\alpha \leq 5$, the inequality (3.27) gives rise to the relation

$$d^3 - 126d^2 + 80d + 120 \leq 0,$$

provided the polynomial on the left side of (3.27) is non-negative⁹. From this it follows that all surfaces X of general type with slope $\alpha \leq 5$ and $m_X = 4$ have degree $d \leq 125$.

2) For complete intersections $(4, a)$ in \mathbb{P}^4 with $a = \frac{d}{4} \geq 2$, we have $K_X = (a - 1)H$ and $\chi(\mathcal{O}_X) = p_g(X) + 1 = \binom{a+3}{4} - \binom{a-1}{4}$. Hence

$$\begin{aligned} K_X^2 &= 4a(a - 1)^2 \\ \chi(\mathcal{O}_X) &= \frac{1}{3}(2a^3 - 3a^2 + 7a - 3) \\ \alpha_X = f(a) &= \frac{12a(a - 1)^2}{2a^3 - 3a^2 + 7a - 3} \longrightarrow 6 \end{aligned}$$

as $a \longrightarrow \infty$. It is easily checked that $f(a)$ is an increasing function of a in the interval $[2, +\infty[$. In particular, $\alpha_X = f(a) > 5$, for all $a \geq 7$. These examples suggest the following questions: besides complete intersections,

- a) are there surfaces of general type in \mathbb{P}^4 which are contained in a quartic 3-fold (and not on one of a smaller degree) and whose slope $\alpha = K^2/\chi > 5$?
- b) is there an infinite sequence $(X_n)_n$ of surfaces of general type in \mathbb{P}^4 with $m_{X_n} = 4$ such that the slopes $\alpha_{X_n} = K_{X_n}^2/\chi(\mathcal{O}_{X_n})$ converge to 6?

4. NUMERICAL INVARIANTS FOR SURFACES WITH $m_X = 2, 3, 5$

This section treats the remaining values of the smallest degree m_X of a hypersurface containing $X \subset \mathbb{P}^4$. As in Section 3, we assume that the Kodaira dimension of X is non-negative.

⁹This is the case for $d \geq 112$.

Consider the extension sequence (1.6) corresponding to $\xi = \delta_X(s) \in H^1(\Theta_X(-K_X - (5 - m)H))$ provided by Lemma 2.2.

We begin with cases $m_X = 2$ or 3.

THEOREM 4.1. *If $m_X = 2$, then*

$$c_2 - K_X^2 \geq \begin{cases} 3H \cdot K_X + 3d, & \text{if } \mathcal{T}_\xi \text{ is Bogomolov semistable} \\ \frac{3}{2}H \cdot K_X + \frac{9}{4}d, & \text{if } \mathcal{T}_\xi \text{ is Bogomolov unstable.} \end{cases}$$

THEOREM 4.2. *If $m_X = 3$, then*

$$c_2 - K_X^2 \geq \begin{cases} 2H \cdot K_X + \frac{4}{3}d, & \text{if } \mathcal{T}_\xi \text{ is Bogomolov semistable} \\ \min\left(K \cdot H + d, \frac{3}{2}K \cdot H + \frac{1}{3}d\right), & \text{if } \mathcal{T}_\xi \text{ is Bogomolov unstable.} \end{cases}$$

We omit the proofs since they follow exactly the same pattern as the one of Theorem 3.1. We only mention, that in the case $m_X = 2$ the only possibility for the Bogomolov filtration is as in Lemma 3.3.

We now turn to the case $m_X = 5$. The sequence (2.3) takes the form

$$H^0(\Theta_{\mathbb{P}^4}|_X(-K_X)) \longrightarrow H^0(\mathcal{N}_X(-K_X)) \xrightarrow{\delta_X} H^1(\Theta_X(-K_X))$$

and we seek an analogue of Lemma 2.2.

LEMMA 4.3. *Let s be a global section of $\mathcal{N}_X(-K_X)$ corresponding to a quintic hypersurface containing X . If $\delta_X(s) = 0$ in $H^1(\Theta_X(-K_X))$, then $d \leq 14$ and $K_X^2 \leq 6\chi(\mathcal{O}_X)$. Furthermore, if X is of general type, then $p_g \leq 2$.*

Proof. The vanishing of $\delta_X(s)$ implies that the section s of $\mathcal{N}_X(-K_X)$ comes from a nonzero section of $\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-K_X)$. Thus $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-K_X)) \neq 0$. This together with the Euler sequence of $\Theta_{\mathbb{P}^4}$ leads to two possibilities:

1. $H^0(\mathcal{O}_X(H - K_X)) \neq 0$,
2. $\ker(H^1(\mathcal{O}_X(-K_X)) \rightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(H - K_X))) \neq 0$.

Both of them tell us that $H \cdot K_X \leq H^2 = d$. Furthermore, the equality holds if and only if $\mathcal{O}_X(K_X) = \mathcal{O}_X(H)$ and then one knows that X must be a complete intersection which is impossible due to the condition $m_X = 5$. Thus we have

$$H \cdot K_X < d. \tag{4.1}$$

This and the double point formula give

$$2(K_X^2 - 6\chi) = d^2 - 10d - 5H \cdot K_X > d^2 - 15d. \tag{4.2}$$

In particular, one obtains that $d \leq 14$, provided $K_X^2 \leq 6\chi$. Thus the first assertion of the lemma is a consequence of the inequality $K_X^2 \leq 6\chi$.

Assume $K_X^2 > 6\chi$. This and the assumption that the Kodaira dimension of X is non-negative implies $\chi \geq 1$. Hence $K_X^2 \geq 7$ and $H \cdot K_X > 0$ tell us that X is of general type.

Observe that the inequality $H \cdot K_X < d$ and the Hodge index give also the upper bound $K_X^2 < d$. Substituting into (4.2), we deduce

$$0 > d^2 - 17d + 12,$$

and hence $d \leq 16$. This and the inequalities $6\chi < K_X^2 < d$ imply

$$K_X^2 \leq 15 \quad \text{and} \quad \chi \leq 2.$$

Furthermore, in the case $K_X^2 = 15$ the degree d must be 16 and the Hodge index $(H \cdot K_X)^2 \geq H^2 K^2 \geq 16 \cdot 15$ implies $H \cdot K_X \geq 16 \geq d$ contrary to (4.1). Thus one has $K_X^2 \leq 14$.

We now examine the remaining possibilities according to two possible values of $\chi = 1$ or 2. If $\chi = 1$, then the double point formula reads

$$d^2 - 10d - 5H \cdot K_X = 2K_X^2 - 12. \quad (4.3)$$

By Bogomolov-Miyaoka-Yau inequality, $K_X^2 \leq 9\chi = 9$. Thus $K_X^2 \in \{7, 8, 9\}$, implying that

$$d^2 - 10d \geq 2 + 5H \cdot K_X \geq 32, \quad (4.4)$$

where the last inequality comes from the Hodge index $(H \cdot K_X)^2 \geq H^2 K_X^2 = d K_X^2 \geq 5 \cdot 7 = 35$ and where the last inequality uses $d \geq 5$, coming from the fact that X is of general type. Hence $d \geq 13$. Using this lower bound in the Hodge Index estimate of $H \cdot K_X$ once again, we obtain $H \cdot K_X \geq 10$. Substituting this in (4.4), implies $d \geq 14$. With the previously derived upper bound $d \leq 16$, we obtain $d = 14, 15, 16$ are the only possible values. A direct check shows that $d = 15$ is incompatible with the double point formula (4.3), while for $d = 14$ and 16, that formula forces $K_X^2 = 9$ and hence, $H \cdot K_X = 10$ and 18, respectively. The latter value contradicts the inequality (4.1), while the former, the Hodge Index inequality.

If $\chi = 2$, then the double point formula becomes

$$d^2 - 10d - 5H \cdot K_X = 2K_X^2 - 24, \quad (4.5)$$

while 13, 14 are the only possible values for K_X^2 . Arguing as in the previous case we obtain the lower bound $H \cdot K_X \geq 9$ which together with (4.5) gives the possibilities $d = 14, 15, 16$ for the degree of X . But then, going back to the lower bound estimate for $H \cdot K_X$, we obtain $H \cdot K_X \geq 14$. Substituting into (4.5) gives $d \geq 15$ and hence, $d = 15$ or 16. The value $d = 15$ is again incompatible with the double point formula (4.5), while for $d = 16$ that formula tells us that $K_X^2 = 14$ and $H \cdot K_X = 16$. But this contradicts the inequality (4.1).

The last statement saying that $p_g \leq 2$, for X of general type, can be seen as follows. Let X_0 be the minimal model of X and set $\sigma : X \rightarrow X_0$ to be a sequence of blow-down maps. Then the canonical divisor K_X can be written as follows

$$K_X = \sigma^* K_{X_0} + D,$$

where D is an effective divisor composed of curves contracted by σ . From the nonvanishing of $H^0(\Theta_X(-K_X)) = H^0(\Theta_X(-\sigma^* K_{X_0} - D))$ it follows that $H^0(\Theta_X(-\sigma^* K_{X_0})) \neq 0$ as well. Running the argument of the first paragraph of the proof for that group and using the fact that $H^1(\mathcal{O}_X(-\sigma^* K_{X_0})) = 0$, we obtain $H^0(\mathcal{O}_X(H - \sigma^* K_{X_0})) \neq 0$.

Next we observe that $(H - \sigma^* K_{X_0}) \neq 0$, since otherwise $H = \sigma^* K_{X_0} = K_X$ and, as it was argued above, this is incompatible with the condition $m_X = 5$.

Once $(H - \sigma^* K_{X_0}) \neq 0$, we take a divisor $\Gamma \in |H - \sigma^* K_{X_0}|$ and identify p_g as follows

$$p_g = h^0(\mathcal{O}_X(\sigma^* K_{X_0})) = h^0(\mathcal{O}_X(H - \Gamma)),$$

i.e., p_g is the dimension of the space of hyperplanes in \mathbb{P}^4 containing Γ . From this it follows that $p_g \leq 3$ and the equality holds if and only if Γ is a line in \mathbb{P}^4 . We claim that this is impossible. Indeed, if Γ is a line then the identity $H = \Gamma + \sigma^* K_{X_0}$ implies

$$1 = \Gamma \cdot H = \Gamma^2 + \Gamma \cdot \sigma^* K_{X_0}. \quad (4.6)$$

Hence $\Gamma \cdot \sigma^* K_{X_0} > 0$ and hence Γ is not in the exceptional divisor D . Therefore $\Gamma \cdot D \geq 0$. But then the identity (4.6) can be rewritten as

$$1 = \Gamma^2 + \Gamma \cdot \sigma^* K_{X_0} = \Gamma^2 + \Gamma \cdot (K_X - D) = (\Gamma^2 + \Gamma \cdot K_X) - \Gamma \cdot D = -2 - \Gamma \cdot D,$$

which is absurd. Thus $p_g \leq 2$ as asserted. \square

Question. Can one enumerate all surfaces in \mathbb{P}^4 with $m_X = 5$ and $\delta_X(s) = 0$?

From now on we set $\xi = \delta_X(s) \in H^1(\Theta_X(-K_X))$ and assume it to be nonzero. The corresponding extension sequence has the form

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0. \quad (4.7)$$

The Bogomolov semistability/instability considerations of the sheaf \mathcal{T}_ξ above give the following.

THEOREM 4.4. *Let X be a surface in \mathbb{P}^4 with $m_X = 5$. Then, either $K_X^2 \leq c_2$, or X is a surface of general type subject to the following properties:*

- i) *The canonical divisor K_X admits a distinguished decomposition $K_X = L + E$, such that L is in the positive cone of $N(X)$ and E is an effective nonzero divisor.*
- ii) $0 < K_X^2 - c_2 \leq \frac{2}{3} L^2 \leq \frac{2}{3} d$.

Proof. We may assume that $K_X^2 > c_2$, since otherwise there is nothing to prove. This assumption and Lemma 4.3 insure the nonvanishing of the cohomology class $\xi = \delta_X(s)$ defined by a section $s \in H^0(\mathcal{N}_X(-K_X))$ arising from a hypersurface of degree 5 containing X . We associate to ξ the extension sequence (4.7) and observe that the assumption $K_X^2 > c_2$ implies that the sheaf \mathcal{T}_ξ in the middle of that sequence is Bogomolov unstable. Let \mathcal{F} be the maximal Bogomolov destabilizing subsheaf of \mathcal{T}_ξ . Observe that the Bogomolov destabilizing property implies that $\mathcal{O}_X(L)$, the determinant of \mathcal{F} , is in the positive cone of $N(X)$.

Putting the inclusion $\mathcal{F} \subset \mathcal{T}_\xi$ together with the extension sequence (4.7) we obtain the diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{F} & & & & \\ & & \downarrow & \searrow \varphi & & & \\ 0 & \longrightarrow & \mathcal{O}_X(-K_X) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X \longrightarrow 0 \end{array} \quad (4.8)$$

where the slanted arrow is the composition of the inclusion together with the epimorphism in (4.7). We claim that \mathcal{F} must be of rank 2 and the morphism φ in the above diagram is

generically an isomorphism. Indeed, if the rank of \mathcal{F} is one, then it is $\mathcal{O}_X(L)$ and φ must be zero, since by the Bogomolov Lemma, Ω_X admits no rank 1 subsheaves of Iitaka dimension 2. But the vanishing of φ implies $\mathcal{O}_X(L) = \mathcal{O}_X(-K_X)$. Hence X is rational with $K_X^2 > 6\chi = 6$. This together with the Hodge index gives a lower bound on the degree of $-K_X$:

$$-K_X \cdot H \geq 5. \quad (4.9)$$

On the other hand, the Riemann-Roch for $\mathcal{O}_X(H)$ yields

$$h^0(\mathcal{O}_X(H)) \geq \frac{d - K_X \cdot H}{2} + 1.$$

Furthermore, $h^0(\mathcal{O}_X(H)) = 5$, since X is not the projection of the Veronese surface from a point outside its secant variety¹⁰. Substituting this in the above inequality we obtain

$$d - K_X \cdot H \leq 8.$$

This and the lower bound (4.9) imply $d = 3$ and $-K_X \cdot H = 5$. Therefore, a general hyperplane section, call it C , is a rational normal cubic in \mathbb{P}^3 and hence C is cut out by quadrics in \mathbb{P}^3 . But then, in view of the isomorphism

$$H^0(\mathcal{I}_X(2)) \cong H^0(\mathcal{I}_C(2)),$$

one obtains quadric hypersurfaces containing X contradicting the assumption $m_X = 5$.

We know now that the rank of \mathcal{F} is 2 and we need to check that the morphism φ in the diagram (4.8) is generically an isomorphism. Assuming that this is not the case and taking the second exterior power of the diagram (4.8), we deduce a nonzero morphism $\mathcal{O}_X(L) \rightarrow \Theta_X$. This implies once again that X must be rational with $K_X^2 > 6\chi = 6$. Thus we are back in the situation considered in the preceding paragraph implying that φ in (4.8) is generically an isomorphism. Hence we are in the position to apply Lemma 2.3; in particular, we obtain the decomposition of the canonical divisor K_X asserted in i) of the theorem. Furthermore, Lemma 2.3, 3) gives us the upper bound

$$L \cdot H \leq H^2 = d.$$

This, by the Hodge Index Theorem, implies

$$L^2 \leq d. \quad (4.10)$$

The part ii) of the theorem is obtained by estimating the second Chern number of \mathcal{T}_ξ from the vertical sequence in the diagram (4.8). Namely, we complete it to the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{T}_\xi \longrightarrow \mathcal{I}_Z(-L) \longrightarrow 0$$

from which we obtain

$$c_2 - K_X^2 = c_2(\mathcal{T}_\xi) = c_2(\mathcal{F}) - L^2 + \deg(Z) \geq c_2(\mathcal{F}) - L^2 \geq \frac{1}{3}L^2 - L^2 = -\frac{2}{3}L^2,$$

where the second inequality is a result of Miyaoka, [24, Remark 4.18]. Hence the first upper bound

$$K_X^2 - c_2 \leq \frac{2}{3}L^2$$

asserted in the part ii) of the theorem. This together with (4.10) gives the second upper bound in ii). \square

¹⁰Such a surface is contained in cubic hypersurfaces and this is contrary to our assumption that the minimal degree of a hypersurface containing X is 5.

PROPOSITION 4.5. *For all surfaces of general type lying on a quintic hypersurface in \mathbb{P}^4 and not on one of a smaller degree, the following assertions hold:*

- 1) *For every rational number $\alpha \neq 6$, there exists $d(\alpha) \in \mathbb{N}$ such that every surface of slope α , has the degree $d \leq d(\alpha)$.*
- 2) *For every rational number $\alpha \neq 6$, all surfaces with slope α and degree $d \leq d(\alpha)$ have the holomorphic Euler characteristic subject to the inequalities*

$$\chi \leq \begin{cases} \frac{5d}{6-\alpha}, & \text{if } \alpha < 6, \\ \frac{d}{3(\alpha-6)}, & \text{if } \alpha > 6. \end{cases}$$

- 3) *The slope of Chern numbers satisfies $\alpha = \frac{K^2}{\chi} < 6 + \frac{1}{2}$, with possible exceptions of surfaces having degree ≤ 37 and holomorphic Euler characteristic ≤ 24 .*

Proof. For $\alpha < 6$ the argument is analogous to the one in the proof of Proposition 3.6. Namely, the Decker-Schreyer polynomial in (1.2) for $m = 5$ has the form

$$P_5(d) = \frac{1}{25} d(d^2 - 40d + 95)$$

and their result gives the lower bound

$$\chi \geq \frac{1}{25} d(d^2 - 40d + 95) \quad (4.11)$$

for χ , the holomorphic Euler characteristic, for every surface of degree $d \geq 18$ contained in a quintic hypersurface and not in one of a smaller degree. Combining this with the double point formula, we obtain

$$\frac{2}{25} (6 - \alpha) d(d^2 - 40d + 95) \leq 2(6 - \alpha) \chi = 12\chi - K^2 = 5H \cdot K_X + 10d - d^2. \quad (4.12)$$

From [15] we also know that the genus g_H of a (smooth) hyperplane section of X is subject to the inequality

$$d + H \cdot K_X = 2g_H - 2 \leq \frac{d^2 + 5d}{5}. \quad (4.13)$$

From this and (4.12), we deduce

$$\frac{1}{25} (6 - \alpha) (d^2 - 40d + 95) \leq 5d.$$

Hence one can explicitly determine a positive integer $d_0(\alpha)$ depending only on α , which is an upper bound for the integer solutions of the above inequality. Setting $d(\alpha) := \max(17, d_0(\alpha))$, we obtain assertion 1) of the proposition in the range $\alpha < 6$.

If $\alpha > 6$, then Theorem 4.4 ii) tells us that $(\alpha - 6)\chi = K^2 - 6\chi \leq \frac{1}{3}d$. Combining this with (4.11) gives the inequality

$$\frac{1}{25} (\alpha - 6) (d^2 - 40d + 95) \leq \frac{(\alpha - 6)\chi}{d} \leq \frac{1}{3}, \quad (4.14)$$

which provides the assertion 1) of the proposition in the range $\alpha > 6$.

The second inequality for χ in 2) is just a restatement of Theorem 4.4, ii), while the first one is obtained by combining (4.12) and the upper bound $5H \cdot K_X \leq d^2$ from (4.13).

For the third part of the proposition which asserts the upper bound for the slope α , we may assume $\alpha > 6$ and go back to the inequality (4.14). For $d \geq 38$, that inequality implies

$$\alpha - 6 \leq \frac{25}{3 \cdot 19} < \frac{1}{2}.$$

For $d \leq 37$, we use the inequality in Theorem 4.4, ii), to deduce

$$(\alpha - 6)\chi = K^2 - 6\chi \leq \frac{1}{3}d \leq \frac{37}{3}.$$

Hence $(\alpha - 6)\chi \leq 12$ and $\alpha < 6 + \frac{1}{2}$, unless $\chi \leq 24$. \square

As a consequence, we obtain the following finiteness result.

COROLLARY 4.6. *The number of families of surfaces in \mathbb{P}^4 of general type with fixed slope $\alpha \neq 6$ and contained in a quintic hypersurface (and not in one of a smaller degree) is at most finite.*

Proof. See the proof of Corollary 3.7. \square

(II) Irregularity of surfaces in \mathbb{P}^4

5. THE IRREGULARITY OF SURFACES LYING ON A SMALL DEGREE HYPERSURFACE

One of the outstanding problems about surfaces in \mathbb{P}^4 is the control of their irregularity. Since the beautiful work of Horrocks and Mumford, [17], which contains a construction of an abelian surface of degree 10 in \mathbb{P}^4 , the irregularity 2 remains the maximal known value for surfaces in \mathbb{P}^4 . Indeed, it is conjectured that no surface of irregularity greater than 2 can be embedded into \mathbb{P}^4 . To our knowledge there is no conceptual reason for this phenomenon.

In our previous work, [26], we were able to show that the irregularity of surfaces in \mathbb{P}^4 is bounded by 3 under a certain precise set of conditions, see [26, Theorem 5.1]. What is perhaps more interesting, is that we have uncovered how to use the cohomological condition $H^1(\Theta_X(-K_X)) \neq 0$ to bound the irregularity of a surface $X \subset \mathbb{P}^4$. In the previous sections we have seen that this non-vanishing condition arises naturally whenever X is contained in a hypersurface of a degree $m \leq 5$. This phenomenon together with the metha-principle formulated in the introduction suggests that the irregularity of surfaces in \mathbb{P}^4 , with the exception of a finite number of families, is bounded by the irregularity of surfaces contained in hypersurfaces of small degree. Guided by this heuristic principle, this section begins the investigation of the irregularity of surfaces in \mathbb{P}^4 contained in a small degree hypersurface.

Our study of irregular surfaces in \mathbb{P}^4 lying on a small degree hypersurfaces has the same unifying theme as before: the extension construction. We consistently interpret a small degree hypersurface containing our surface as an extension sequence of sheaves on X . Let m be the smallest degree of a hypersurface containing a surface $X \subset \mathbb{P}^4$ and let V_m be such a hypersurface. We recall that \mathcal{N}_X (resp. \mathcal{N}_X^*) denotes the normal (resp. conormal) bundle of X in \mathbb{P}^4 and one has the following identifications

$$\mathcal{J}_X/\mathcal{J}_X^2 = \mathcal{N}_X^* \cong \det(\mathcal{N}^*) \otimes \mathcal{N}_X = \mathcal{N}_X(-K_X - 5H),$$

where \mathcal{J}_X is the ideal sheaf of X in \mathbb{P}^4 and the second identification is due to the rank of \mathcal{N}_X^* being two. This leads to the non-vanishing

$$H^0(\mathcal{N}_X(-K_X - (5 - m)H)) = H^0(\mathcal{N}_X^*(mH)) \neq 0$$

already encountered in (2.2). Thus, we associate to V_m a nonzero global section, denoted by s , of the twisted normal bundle $\mathcal{N}_X(-K_X - (5 - m)H)$. Our approach consists of using this section to build up an appropriate extension sequence.

In the context of the irregularity, there are two lines of thinking. The first one is to use the coboundary map δ_X in the normal exact sequence for $X \subset \mathbb{P}^4$ to produce the cohomology class $\xi = \delta_X(s) \in H^1(\Theta_X(-K_X - (5 - m)H))$ and then, to view it, via the natural identification

$$H^1(\Theta_X(-K_X - (5 - m)H)) \cong \text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X - (5 - m)H)),$$

as an extension

$$0 \longrightarrow \mathcal{O}_X(-K_X - (5 - m)H) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0.$$

This was the idea exploited in [26] and it produces satisfactory results provided that X is of Albanese dimension 2, *i.e.*, the image of the Albanese morphism of X is of dimension 2. However, if X fibers over a curve B of genus $g(B) = q(X)$, the method fails.

This brings us to the second way of associating an extension sequence to the section $s \in H^0(\mathcal{N}_X(-K_X - (5 - m)H))$. Namely, we simply take the Koszul sequence associated to s to obtain

$$0 \longrightarrow \mathcal{O}_X(K_X + (5 - m)H) \xrightarrow{s} \mathcal{N}_X \xrightarrow{\wedge^s} \mathcal{J}_Z(mH) \longrightarrow 0, \quad (5.1)$$

where $Z \subset X$ is the scheme of zeros of s and \mathcal{J}_Z its ideal sheaf. This is of course very classical and yet efficient in addressing the irregularity problem, provided we have a good control of the subscheme Z . Let us explain the main ingredients of this approach as well as set up the stage for more technical considerations in the subsequent sections.

The extension (5.1) fails to be exact precisely when $Z = (s = 0)$ has divisorial part. If this is the case, let Z_1 be the divisorial part and Z_0 be the residual part of Z_1 in Z . If s_1 is a section of $\mathcal{O}_X(Z_1)$ defining Z_1 , then $s = s_1 s_0$, where s_0 is a section of $\mathcal{N}_X(-K_X - (5 - m)H - Z_1)$ whose zero-locus is Z_0 , a 0-dimensional subscheme of X . We now have the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + (5 - m)H + Z_1) \xrightarrow{s_0} \mathcal{N}_X \xrightarrow{\wedge^{s_0}} \mathcal{J}_{Z_0}(mH - Z_1) \longrightarrow 0, \quad (5.2)$$

where \mathcal{J}_{Z_0} is the ideal sheaf of Z_0 .

At this stage we bring in the irregularity. The main point in relating the irregularity of X to the extension sequence (5.2) is the following general fact.

LEMMA 5.1. *If $X \subset \mathbb{P}^n$ is a complex projective manifold of dimension bigger than 1, then*

$$H^0(\Omega_X) \cong H^1(\mathcal{N}_X^*),$$

where Ω_X (resp. \mathcal{N}_X^*) is the cotangent (resp. conormal) bundle of X .

Proof. From the conormal exact sequence

$$0 \longrightarrow \mathcal{N}_X^* \longrightarrow \Omega_{\mathbb{P}^n}|_X \longrightarrow \Omega_X \longrightarrow 0$$

of $X \subset \mathbb{P}^n$ we obtain

$$0 \longrightarrow H^0(\Omega_X) \longrightarrow H^1(\mathcal{N}_X^*) \longrightarrow H^1(\Omega_{\mathbb{P}^n|X}) \xrightarrow{r} H^1(\Omega_X)$$

where the injectivity on the left is the vanishing of $H^0(\Omega_{\mathbb{P}^n|X})$. The asserted isomorphism follows from the injectivity of the homomorphism $r : H^1(\Omega_{\mathbb{P}^n|X}) \rightarrow H^1(\Omega_X)$. To establish it we use the dual of the Euler sequence and the assumption $\dim(X) > 1$ to deduce $H^1(\Omega_{\mathbb{P}^n|X}) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}$. At the same time, we have the linear map

$$\mathbb{C} \cong H^1(\Omega_{\mathbb{P}^n}) \xrightarrow{i^*} H^1(\Omega_X)$$

defined by the pullback i^* , where $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion morphism. This linear map factors through $H^1(\Omega_{\mathbb{P}^n|X})$ to give rise to the commutative diagram

$$\begin{array}{ccc} H^1(\Omega_{\mathbb{P}^n}) & \xrightarrow{i^*} & H^1(\Omega_X) \\ & \searrow & \nearrow r \\ & H^1(\Omega_{\mathbb{P}^n|X}) & \end{array}$$

and the injectivity of r follows from the injectivity of i^* . The latter is injective, since it sends the generator $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^1(\Omega_{\mathbb{P}^n})$ to the class of a hyperplane section of X . \square

In case X is a surface, the isomorphism in Lemma 5.1 and the Serre duality yield the identification

$$H^0(\Omega_X) \cong H^1(\mathcal{N}_X^*) \cong H^1(\mathcal{N}_X(K_X))^*.$$

Now we can see that the extension sequence (5.2) tensored with $\mathcal{O}_X(K_X)$ is tied to the irregularity of X via the exact sequence

$$H^1(\mathcal{O}_X(2K_X + (5-m)H + Z_1)) \xrightarrow{s_0} H^1(\mathcal{N}_X(K_X)) \xrightarrow{\wedge^{s_0}} H^1(\mathcal{J}_{Z_0}(K_X + mH - Z_1)). \quad (5.3)$$

Thus the problem of computing or bounding the irregularity comes down to the understanding of the cohomology groups $H^1(\mathcal{O}_X(2K_X + (5-m)H + Z_1))$ and $H^1(\mathcal{J}_{Z_0}(K_X + mH - Z_1))$. This in turn depends on controlling the subscheme Z and the decomposition

$$Z = Z_1 + Z_0.$$

The subscheme Z is related to the singular locus of the hypersurface V_m which was used to define the section s . Let us spell out this relationship.

The normal sequence of $X \subset \mathbb{P}^4$ and the Euler sequence of $\Theta_{\mathbb{P}^4}$ give rise to the surjective morphism

$$H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X(H) \longrightarrow \mathcal{N}_X.$$

This together with the section $s \in H^0(\mathcal{N}_X^*(mH))$ —corresponding to V_m —yields the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X(H) & & \\ \downarrow & \searrow & \\ \mathcal{N}_X & \xrightarrow{s \wedge} & \mathcal{J}_Z(mH) \end{array} \quad (5.4)$$

where the composite (slanted) arrow is given by the partial differentiation of a homogeneous polynomial defining V_m . Thus, denoting by \mathfrak{I}_{V_m} the sheaf of ideals defined by the Jacobian ideal of V_m , the diagram (5.4) tells us that we have the inclusion

$$\mathcal{J}_Z \supset \mathfrak{I}_{V_m} \otimes \mathcal{O}_X$$

or, equivalently, that Z is a subscheme of the scheme-theoretic intersection of the singular locus $\text{Sing}(V_m)$ with X . So it is reasonable to expect that one could control Z and hence, the cohomology groups in (5.3), for small values of m . As an easy illustrative example of the above ideas, let us work out the case $m = 2$.

6. IRREGULAR SURFACES ON HYPERSURFACES OF DEGREE 2

The following theorem, no doubt, is well-known to experts, see *e.g.*, [30], p. 152.

THEOREM 6.1. *If $X \subset \mathbb{P}^4$ is a smooth surface lying on a quadric hypersurface V_2 , then the irregularity of X vanishes.*

Proof. Suppose that $X \subset V_2$. We may assume that V_2 is singular, since otherwise X is a complete intersection and we are done by Lefschetz hyperplane theorem. The singular locus $\text{Sing}(V_2)$ is either a single point, the vertex p of V_2 , or a line, L .

The case $\text{Sing}(V_2) = \{p\}$. The extension sequence (5.1) takes the form

$$0 \longrightarrow \mathcal{O}_X(2K_X + 3H) \xrightarrow{s} \mathcal{N}_X(K_X) \longrightarrow \mathcal{J}_Z(K_X + 2H) \longrightarrow 0$$

where Z is either empty or p . Thus the sequence of cohomology groups (5.3) becomes

$$H^1(\mathcal{O}_X(2K_X + 3H)) \xrightarrow{s} H^1(\mathcal{N}_X(K_X)) \longrightarrow H^1(\mathcal{J}_Z(K_X + 2H)).$$

Thus the irregularity is bounded as follows.

$$q(X) = h^1(\mathcal{N}_X(K_X)) \leq h^1(\mathcal{O}_X(2K_X + 3H)) + h^1(\mathcal{J}_Z(K_X + 2H)).$$

From [29] it follows that the divisor $K_X + 3H$ is very ample. Hence, by the Kodaira vanishing and the Serre duality, we obtain $h^1(\mathcal{O}_X(2K_X + 3H)) = 0$.

We turn now toward $h^1(\mathcal{J}_Z(K_X + 2H))$. If $Z = 0$, then

$$H^1(\mathcal{J}_Z(K_X + 2H)) = H^1(\mathcal{O}_X(K_X + 2H)) \stackrel{(SD)}{\cong} H^1(\mathcal{O}_X(-2H))^* = 0,$$

and hence, $q(X) = 0$ in this case. If $Z = p$, then the non-vanishing of $H^1(\mathcal{J}_p(K_X + 2H))$ means that p is a base point of $\mathcal{O}_X(K_X + 2H)$, but this is ruled out by [29, Theorem 1, (i)]. Hence $H^1(\mathcal{J}_p(K_X + 2H)) = 0$. This completes the proof of the theorem when $\text{Sing}(V_2)$ is a point.

The case $\text{Sing}(V_2) = L$. In this case V_2 is a singular rational scroll over a smooth conic C lying in a plane Π complementary to the line L . It is ruled by the one parameter family of planes $\{P_t\}_{t \in C}$, where the plane P_t is the span of t and L . There are two cases to consider according to whether or not the line L is contained in X .

- If $L \subset X$, then the projection from L defines the morphism

$$\varphi : X \longrightarrow C \cong \mathbb{P}^1$$

with the fibre F_t over a point $t \in C$ being the component of the intersection $P_t \cdot X$ complementary to L . This implies that the geometric ingredients Z_1 and Z_0 in the cohomology sequence (5.3) are the line L and the empty set, respectively. Hence that sequence takes the form

$$H^1(\mathcal{O}_X(2K_X + 3H + L)) \longrightarrow H^1(\mathcal{N}_X(K_X)) \longrightarrow H^1(\mathcal{O}_X(K_X + 2H - L)).$$

We claim that the cohomology groups $H^1(\mathcal{O}_X(2K_X + 3H + L))$ and $H^1(\mathcal{O}_X(K_X + 2H - L))$ vanish, since the divisors $2H - L$ and $K_X + 3H + L$ are both ample. Indeed, writing $2H - L = H + (H - L)$ and using the fact that the linear system $|H - L|$ is base point free (the linear system defines the morphism φ), we deduce that $2H - L$ is very ample. The ampleness of $K_X + 3H + L$ is checked easily using the very ampleness of $K_X + 3H$ and the Nakai-Moishezon criterion, see [16].

- If $L \not\subset X$, then the sequence (5.1) associated to V_2 is exact and has the form

$$0 \longrightarrow \mathcal{O}_X(K_X + 3H) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{J}_Z(2H) \longrightarrow 0, \quad (6.1)$$

where Z is the 0-dimensional subscheme of X , the scheme-theoretic intersection of X and the line L . This gives the relations

$$d^2 - 4d - \deg(Z) = 4(g_H - 1) \text{ and } \deg(Z) < d. \quad (6.2)$$

We will now calculate $\deg(Z)$ using the geometry of the singular scroll V_2 containing X . Namely, we go back to the plane Π containing the conic C and complementary to L and consider a hyperplane $\text{Span}(L, \Lambda)$ in \mathbb{P}^4 spanned by L and a general line in $\Lambda \subset \Pi$.

The hyperplane $\text{Span}(L, \Lambda)$ intersects V_2 in the union of two planes P_t , where $t \in \Lambda \cdot C$. Hence $H = \text{Span}(L, \Lambda) \cdot X = 2F$, where F is the class of the divisor $P_{t'} \cdot X$, for a closed point $t' \in C$. From this it follows that $d = 4F^2$, and, by adjunction,

$$2(g_H - 1) = 2F \cdot K_X + 4F^2 = 2\deg(\omega_F) + 2F^2 = 2\deg(\omega_F) + \frac{d}{2},$$

where ω_F is the dualizing sheaf of F . On the other hand, since F is a plane divisor, its dualizing sheaf is subject to

$$\deg(\omega_F) = d_F(d_F - 3) = \frac{d}{2} \left(\frac{d}{2} - 3 \right) = \frac{1}{4} d(d - 6).$$

Substituting into the previous identity, we have

$$2(g_H - 1) = \frac{1}{2} d(d - 6) + \frac{d}{2} = \frac{1}{2} d(d - 5).$$

Putting it together with the identity in (6.2), we obtain $\deg(Z) = d$ which contradicts the inequality in (6.2). \square

We close this ‘warm up’ section with a simple general observation concerning the sheaf $\mathcal{J}_{Z_0}(mH - Z_1)$ in the sequence (5.2). This observation will be important in studying irregular surfaces lying on cubic hypersurfaces.

LEMMA 6.2. *The sheaf $\mathcal{J}_{Z_0}((m-1)H - Z_1)$ (see the sequence (5.2) for notation), is generated by global sections outside the 0-dimensional subscheme Z_0 . Furthermore, $h^0(\mathcal{J}_{Z_0}((m-1)H - Z_1)) \geq 5$, unless the hypersurface V_m is a cone over a surface in \mathbb{P}^3 .*

Proof. From the sequence (5.2) tensored with $\mathcal{O}_X(-H)$ it follows that $\mathcal{J}_{Z_0}((m-1)H - Z_1)$ is the quotient of $\mathcal{N}_X(-H)$ which is globally generated. Hence the first statement of the lemma.

For the second statement, we use the diagram (5.4) tensored with $\mathcal{O}_X(-H)$. The slanted arrow is the morphism

$$H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X \longrightarrow \mathcal{J}_Z((m-1)H)$$

which factors through $\mathcal{J}_{Z_0}((m-1)H - Z_1)$ and induces, at the level of global sections, the linear map

$$\partial(f)|_X : H^0(\mathcal{O}_X(H))^* \longrightarrow H^0(\mathcal{J}_{Z_0}((m-1)H - Z_1)), \quad (6.3)$$

where f is a homogeneous polynomial defining V_m . The map $\partial(f)|_X$ takes a vector $v \in H^0(\mathcal{O}_X(H))^*$ to the partial derivative $\partial_v(f)|_X$ restricted to X and then divides it by a section defining Z_1 . From this it follows that the kernel of $\partial(f)|_X$ consists of vectors $v \in H^0(\mathcal{O}_X(H))^*$ for which $\partial_v(f)$ is a homogeneous polynomial of degree $m-1$ vanishing on X . Since by definition, m is the least degree of such polynomials, we deduce that $\partial_v(f) = 0$ in $\text{Sym}^{m-1}(H^0(\mathcal{O}_X(H)))$ or, equivalently,

$$f \in \text{Sym}^{m-1}(v^\perp),$$

where $v^\perp = \{l \in H^0(\mathcal{O}_X(H)) \mid l(v) = 0\}$. In particular, for $v \neq 0$, the above means that the point $[v] \in \mathbb{P}(H^0(\mathcal{O}_X(H))^*) = \mathbb{P}^4$ is the vertex of the cone over the surface in $\mathbb{P}((v^\perp)^*) \cong \mathbb{P}^3$ defined by f , viewed as a homogeneous polynomial on $\mathbb{P}((v^\perp)^*)$. Hence, unless V_m is a cone over a surface in \mathbb{P}^3 , the operator $\partial(f)|_X$ in (6.3) is injective and, therefore,

$$h^0(\mathcal{J}_{Z_0}((m-1)H - Z_1)) \geq h^0(\mathcal{O}_X(H)) \geq 5$$

as asserted in the lemma. □

7. IRREGULAR SURFACES ON HYPERSURFACES OF DEGREE 3

The main result of this section is the following characterization of irregular surfaces contained in a cubic hypersurface in \mathbb{P}^4 . Results concerning surfaces contained in a cubic hypersurface in \mathbb{P}^4 can be found, *e.g.*, in [21] and [30].

THEOREM 7.1. *If $X \subset \mathbb{P}^4$ is a smooth irregular surface contained in a cubic hypersurface V_3 , then X is an elliptic scroll of degree 5. Moreover, a general cubic hypersurface containing X is a Segre cubic and its ten singular points lie on X .*

As we have already explained, our main tool is the exact sequence (5.2) which, for $m = 3$, takes the form

$$0 \longrightarrow \mathcal{O}_X(K_X + 2H + Z_1) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{J}_{Z_0}(3H - Z_1) \longrightarrow 0. \quad (7.1)$$

The corresponding cohomological sequence (5.3) controlling the irregularity of X becomes

$$H^1(\mathcal{O}_X(2K_X + 2H + Z_1)) \longrightarrow H^1(\mathcal{N}_X(K_X)) \longrightarrow H^1(\mathcal{J}_{Z_0}(K_X + 3H - Z_1)). \quad (7.2)$$

To analyse the cohomology group on the left in the above sequence we will need the following.

PROPOSITION 7.2. *If $H^1(\mathcal{O}_X(2K_X + 2H)) \neq 0$, then X is an elliptic scroll of degree 5.*

Proof. Assume that $H^1(\mathcal{O}_X(2K_X + 2H))$ does not vanish. Using the Serre duality and [29], we deduce that the line bundle $\mathcal{O}_X(K_X + 2H)$ is base point free but not big. Since $\mathcal{O}_X(K_X + 2H)$ is not trivial¹¹, we deduce that the morphism defined by $\mathcal{O}_X(K_X + 2H)$ maps X onto a curve. In other words, there is a morphism $\varphi : X \rightarrow B$ with connected fibres onto a smooth curve B , and a base point free line bundle $\mathcal{O}_B(D)$ on B such that $\mathcal{O}_X(K_X + 2H) = \varphi^*\mathcal{O}_B(D)$. This implies that

$$K_X + 2H = \deg(D)F,$$

where F is the class of a general fibre of φ . Taking the intersection with F on both sides of the above identity implies that $H \cdot F = 1$. This means that the fibres of φ are lines and that X is a minimal ruled surface embedded into \mathbb{P}^4 by $\mathcal{O}_X(H)$ as a scroll with irregularity $q = q(X) = g(B)$, the genus of B . It is well known that the only irregular scroll in \mathbb{P}^4 is an elliptic scroll of degree 5, see Lemma 7.8. \square

Next we turn to the group on the right of the sequence (7.2).

LEMMA 7.3. *If V_3 is not a cone and Z_1 , the divisorial part in (7.2), is non-zero, then $H^1(\mathcal{J}_{Z_0}(K_X + 3H - Z_1)) = 0$.*

Proof. Assume that $H^1(\mathcal{J}_{Z_0}(K_X + 3H - Z_1)) \neq 0$. From the identification

$$H^1(\mathcal{J}_{Z_0}(K_X + 3H - Z_1))^* \cong \text{Ext}^1(\mathcal{J}_{Z_0}(3H - Z_1), \mathcal{O}_X),$$

the supposed nonvanishing is interpreted as a nontrivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z_0}(3H - Z_1) \longrightarrow 0. \quad (7.3)$$

Tensoring it with $\mathcal{O}_X(-H)$, we obtain

$$H^0(\mathcal{E}(-H)) \cong H^0(\mathcal{J}_{Z_0}(2H - Z_1)). \quad (7.4)$$

This and Lemma 6.2 imply

$$h^0(\mathcal{E}(-H)) \geq 5. \quad (7.5)$$

This is going to play the role of a destabilizing condition for \mathcal{E} . Namely, let t be a nonzero global section of $\mathcal{E}(-H)$. It gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{J}_{Z'}(H - A - Z_1) \longrightarrow 0, \quad (7.6)$$

¹¹Otherwise $\mathcal{O}_X(K_X) = \mathcal{O}_X(-2H)$ and hence $q(X) = 0$.

where A is the divisorial part of the zero locus of t and Z' is the 0-dimensional residual part of $(t = 0)$. We proceed now to analyse the above sequence according to the dimension of the linear system $|A|$.

If $h^0(\mathcal{O}_X(A)) \leq 2$, then (7.5) implies $h^0(\mathcal{J}_{Z'}(H - A - Z_1)) \geq 3$. Since $Z_1 \neq 0$, it follows that Z_1 is a line and $A = 0$. But then, $h^0(\mathcal{J}_{Z'}(H - A - Z_1)) = h^0(\mathcal{J}_{Z'}(H - Z_1)) = 3$ and the sequence (7.6), together with the estimate (7.5), imply

$$5 \leq h^0(\mathcal{E}(-H)) \leq h^0(\mathcal{O}_X) + h^0(\mathcal{J}_{Z'}(H - Z_1)) = 1 + 3 = 4,$$

an obvious contradiction. Thus $h^0(\mathcal{O}_X(A)) \geq 3$.

Combining the sequence (7.6) with the defining sequence (7.3) tensored with $\mathcal{O}_X(-H)$, we obtain the nonzero morphism

$$\mathcal{O}_X(A) \longrightarrow \mathcal{J}_{Z_0}(2H - Z_1).$$

Since this morphism can not be an isomorphism¹², it is given by a nonzero section $e \in H^0(\mathcal{J}_{Z_0}(2H - Z_1 - A))$ vanishing on the nonzero divisor

$$E = (e = 0) \in |2H - Z_1 - A|.$$

In particular, the image of the morphism $H^0(\mathcal{O}_X(A)) \xrightarrow{e} H^0(\mathcal{J}_{Z_0}(2H - Z_1))$ consists of global sections of $\mathcal{J}_{Z_0}(2H - Z_1)$ vanishing on E . In view of the global generation of $\mathcal{J}_{Z_0}(2H - Z_1)$ outside Z_0 , see Lemma 6.2, it follows that $e H^0(\mathcal{O}_X(A))$ is a proper subspace of $H^0(\mathcal{J}_{Z_0}(2H - Z_1))$. Combining this and the isomorphism (7.4), we deduce that the image of $H^0(\mathcal{O}_X(A))$ under the monomorphism in (7.6) is a proper subspace of $H^0(\mathcal{E}(-H))$. Hence $H^0(\mathcal{J}_{Z'}(H - Z_1 - A)) \neq 0$.

Let $D = H - Z_1 - A$. By the above, it is an effective divisor and $h^0(\mathcal{O}_X(H - D)) = h^0(\mathcal{O}_X(Z_1 + A)) \geq 3$. This tells us that either $D = 0$ or it is a line and the above inequality must be an equality. The first possibility is equivalent to $A = H - Z_1$ and $\mathcal{J}_{Z'}(H - Z_1 - A) = \mathcal{O}_X$. This implies $h^0(\mathcal{E}(-H)) \leq 4$ which contradicts (7.5). If $D = H - Z_1 - A$ is a line, then the estimate

$$h^0(\mathcal{O}_X(H - Z_1 - D)) = h^0(\mathcal{O}_X(A)) \geq 3$$

implies that $Z_1 + D$ is a line. But since $Z_1 \neq 0$, this is impossible. \square

The above lemma implies the following.

LEMMA 7.4. *If V_3 is not a cone and the divisorial part Z_1 in (7.2) is nonzero, then X is a regular surface.*

Proof. From Lemma 7.3 and (7.2) it follows that the irregularity of X is controlled by the group $H^1(\mathcal{O}_X(2K_X + 2H + Z_1))$. This group is related to the group $H^1(\mathcal{O}_X(2K_X + 2H))$ considered in Proposition 7.2 via the obvious exact sequence

$$H^1(\mathcal{O}_X(2K_X + 2H)) \longrightarrow H^1(\mathcal{O}_X(2K_X + 2H + Z_1)) \longrightarrow H^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)), \quad (7.7)$$

the understanding of which requires to have a good grasp of Z_1 . This is the case, since we know that Z_1 is contained in the 1-dimensional part of the singular locus $\text{Sing}(V_3)$. For a cubic hypersurface, which is not a cone over a cubic surface, the 1-dimensional part of its singular locus is known to be

¹²Otherwise the extension sequence (7.3) is trivial.

- (i) a line,
- (ii) a conic (possibly singular),
- (iii) a rational normal curve of degree 4 in \mathbb{P}^4 .

Thus Z_1 is one of the above possibilities and we analyse each of them separately.

Case (i) — Z_1 is a line. In the exact sequence (7.7), the degree of the line bundle appearing in the group $H^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1))$ satisfies

$$Z_1 \cdot (2K_X + 2H + Z_1) = Z_1 \cdot K_X + 2 + Z_1 \cdot (K_X + Z_1) = Z_1 \cdot K_X + 2 - 2 = Z_1 \cdot K_X = -Z_1^2 - 2.$$

If $Z_1^2 \leq -1$, it follows that $h^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)) = 0$. Combining this and (7.7), we obtain

$$h^1(\mathcal{O}_X(2K_X + 2H + Z_1)) \leq h^1(\mathcal{O}_X(2K_X + 2H))$$

Moreover, from Proposition 7.2, we know that $h^1(\mathcal{O}_X(2K_X + 2H)) \neq 0$ only if X is an elliptic scroll. But the only lines on such a surface X are the rulings. Therefore $Z_1^2 \leq -1$ implies that $h^1(\mathcal{O}_X(2K_X + 2H))$ and hence $h^1(\mathcal{O}_X(2K_X + 2H + Z_1))$ are both equal to zero. Thus we may assume $Z_1^2 = 0$, since X is obviously regular if $Z_1^2 > 0$. Hence X , if irregular, is a scroll. The proof of Proposition 7.2 tells us that it must be an elliptic scroll of degree $d = 5$. In addition, computing the second Chern number of \mathcal{N}_X from the exact sequence (7.1), we obtain

$$25 = d^2 = (K_X + 2H + Z_1)(3H - Z_1) + \deg(Z_0) = 3d + 3 + \deg(Z_0) = 18 + \deg(Z_0)$$

and hence

$$\deg(Z_0) = 7. \tag{7.8}$$

We decompose the 0-dimensional subscheme Z_0 , the zero locus of the section s_0 in (7.1), into two parts,

$$Z_0 = Z_0^1 + Z_0', \tag{7.9}$$

where Z_0^1 is the part of Z_0 lying on the line Z_1 and the residual subscheme Z_0' is supported on singular points of V_3 belonging to $X \setminus Z_1$.

To estimate the degree of Z_0^1 we use the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-kZ_1) \longrightarrow \mathcal{J}_{Z_0^1} \longrightarrow \mathcal{O}_{kZ_1}(-Z_0^1) \longrightarrow 0,$$

where $k \geq 1$ is the smallest multiple of Z_1 containing the subscheme Z_0^1 . Tensoring with $\mathcal{O}_X(2H - Z_1)$ and using the fact that the linear system $|\mathcal{J}_{Z_0^1}(2H - Z_1)|$ has at most a 0-dimensional base locus, see Lemma 6.2, we deduce that $h^0(\mathcal{O}_{kZ_1}(-Z_0^1) \otimes \mathcal{O}_X(2H - Z_1)) > 0$. From this and $Z_1^2 = 0$, we obtain

$$0 \leq \deg(2H|_{Z_1} - Z_0^1) = 2 - \deg(Z_0^1)$$

or, equivalently, $\deg(Z_0^1) \leq 2$. This together with (7.8) tells us that the part Z_0' of the decomposition (7.9) has degree at least 5. In particular, V_3 must have singular points lying on X and outside the line Z_1 . However, this is impossible. Indeed, let p be a singular point of V_3 lying on $X \setminus Z_1$. Then the plane P spanned by Z_1 and p must be contained in V_3 . Now consider the pencil of hyperplanes $\{V(t) \mid t \in \mathbb{P}^1\}$ containing P . Each $V(t)$ intersects

V_3 along $P \cup Q_t$, where Q_t is a quadric surface in $V(t)$ passing through Z_1 and the point p . The hyperplane sections $H_t = V(t) \cdot X$ are reducible and have the form

$$H_t = V(t) \cdot X = Z_1 + \Gamma_t,$$

where Γ_t is the divisor on X residual to Z_1 . Furthermore, Γ_t has degree 4 and is contained in Q_t . One sees immediately that Γ_t can not be irreducible, since otherwise it is a smooth elliptic curve in Q_t and its intersection with the ruling Z_1 must be 2, but on X the curve Γ_t is a section of X and $\Gamma_t \cdot Z_1 = 1$. Hence $\Gamma_t = \Gamma_0 + L_t$, where Γ_0 is a smooth plane cubic, the fixed part of the pencil $\{H_t\}$, and L_t is a ruling of X . This means that the rulings L_t of X are rationally equivalent and this is clearly impossible. This completes the proof of the lemma in the case (i).

Case (ii) — Z_1 is a conic. If Z_1 is a smooth conic, then the argument is as in the previous case: we compute the degree

$$Z_1 \cdot (2K_X + 2H + Z_1) = Z_1 \cdot K_X + 4 + Z_1 \cdot (K_X + Z_1) = Z_1 \cdot K_X + 4 - 2 = Z_1 \cdot K_X + 2 = -Z_1^2$$

and obtain $h^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)) = 0$, unless X is regular. Hence

$$h^1(\mathcal{O}_X(2K_X + 2H + Z_1)) \leq h^1(\mathcal{O}_X(2K_X + 2H)),$$

with the conclusion, in view of Proposition 7.2, that the irregular surface X must be an elliptic scroll. But such a surface can not contain conics.

If Z_1 is singular, then $Z_1 = L_1 + L_2$, the sum of two lines L_i , $i = 1, 2$. The cohomology group $H^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1))$ fits into the following exact sequence

$$H^1(\mathcal{O}_{L_1}(2K_X + 2H + L_1)) \rightarrow H^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)) \rightarrow H^1(\mathcal{O}_{L_2}(2K_X + 2H + L_2)) \quad (7.10)$$

and we continue as in the case (i). Namely, we compute the degree of the line bundle $\mathcal{O}_{L_2}(2K_X + 2H + L_2)$:

$$L_2 \cdot (2K_X + 2H + Z_1) = L_2 \cdot K_X + L_2 \cdot L_1 = -L_2^2 + L_2 \cdot L_1 - 2.$$

This implies that $h^1(\mathcal{O}_{L_2}(2K_X + 2H + L_2)) = 0$, unless $L_1 = L_2$ and $L_1^2 = 0$. The latter condition means that X is a scroll and, by the proof of Proposition 7.2, see also Lemma 7.8, it must be an elliptic scroll of degree $d = 5$. Computing the second Chern number of \mathcal{N}_X from (7.1), we obtain

$$\begin{aligned} 25 = d^2 &= (K_X + 2H + Z_1)(3H - Z_1) \\ &= (K_X + 2H + 2L_1)(3H - 2L_1) + \deg(Z_0) = 3d + 6 + \deg(Z_0) = 21 + \deg(Z_0). \end{aligned}$$

Hence $\deg(Z_0) = 4$ and from here on we repeat the argument from the proof of case (i).

Thus we must have $H^1(\mathcal{O}_{L_2}(2K_X + 2H + L_2)) = 0$. This and (7.10) imply

$$h^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)) \leq h^1(\mathcal{O}_{L_1}(2K_X + 2H + L_1))$$

which puts us back into the situation of the case of the line in (i).

Case (iii) — Z_1 is a rational normal curve of degree 4. This case is very special since a rational normal curve C of degree 4 that lies in the singular locus of V_3 forces the cubic hypersurface V_3 to be the secant variety of C . Hence $\text{Sing}(V_3) = C$ and the Jacobian

ideal \mathcal{I}_{V_3} is equal to the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^4}$ of C in \mathbb{P}^4 . From this, the exact sequence (7.1)—for a smooth surface X contained in V_3 and passing through C —becomes

$$0 \longrightarrow \mathcal{O}_X(2H + K_X + C) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{O}_X(3H - C) \longrightarrow 0,$$

It is now easy to see that X is regular. Indeed, the cohomological sequence controlling the irregularity q of X ,

$$H^1(\mathcal{O}_X(2H + 2K_X + C)) \longrightarrow H^1(\mathcal{N}_X(K_X)) \longrightarrow H^1(\mathcal{O}_X(3H - C + K_X)),$$

tells us that

$$q = h^1(\mathcal{N}_X(K_X)) \leq h^1(\mathcal{O}_X(2H + 2K_X + C)),$$

since the line bundle $\mathcal{O}_X(3H - C)$ is ample.

The group $H^1(\mathcal{O}_X(2H + 2K_X + C))$ is computed by the exact sequence

$$H^1(\mathcal{O}_X(2H + 2K_X)) \longrightarrow H^1(\mathcal{O}_X(2H + 2K_X + C)) \longrightarrow H^1(\mathcal{O}_C(2H + 2K_X + C)).$$

The group on the right is zero unless $C^2 \geq 6$ and this of course is impossible on an irregular surface. If $h^1(\mathcal{O}_C(2H + 2K_X + C)) = 0$, then $h^1(\mathcal{O}_X(2H + 2K_X + C)) \leq h^1(\mathcal{O}_X(2H + 2K_X))$ and Proposition 7.2 tells us that if $h^1(\mathcal{O}_X(2H + 2K_X)) \neq 0$, then X must be a an elliptic scroll. But of course C can not lie on such a surface. Therefore, $h^1(\mathcal{O}_X(2H + 2K_X))$ and hence $h^1(\mathcal{O}_X(2H + 2K_X + C))$ are both equal to zero. This completes the proof of the lemma. \square

We know now that if V_3 is not a cone and contains an irregular surface X , then the exact sequence (7.1) must have the form

$$0 \longrightarrow \mathcal{O}_X(K_X + 2H) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{I}_Z(3H) \longrightarrow 0, \quad (7.11)$$

where Z is a 0-dimensional scheme, the intersection of X with the singular locus of V_3 . Hence, the cohomology sequence controlling the irregularity of X becomes

$$H^1(\mathcal{O}_X(2K_X + 2H)) \longrightarrow H^1(\mathcal{N}_X(K_X)) \longrightarrow H^1(\mathcal{I}_Z(K_X + 3H)).$$

LEMMA 7.5. *If X is irregular, then $H^1(\mathcal{I}_Z(K_X + 3H)) = 0$.*

Proof. Assume $H^1(\mathcal{I}_Z(K_X + 3H)) \neq 0$. Using our approach, we interpret this nonvanishing as a nontrivial extension sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z(3H) \longrightarrow 0. \quad (7.12)$$

Tensoring it with $\mathcal{O}_X(-H)$, we obtain

$$H^0(\mathcal{E}(-H)) \cong H^0(\mathcal{I}_Z(2H)).$$

Combining this and Lemma 6.2 gives

$$h^0(\mathcal{E}(-H)) \geq 5. \quad (7.13)$$

From here on we proceed as in the proof of Lemma 7.3 to obtain a ‘destabilizing’ sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{I}_{Z'}(H - A) \longrightarrow 0. \quad (7.14)$$

Case $h^0(\mathcal{O}_X(A)) \leq 2$. In this case (7.13) implies $h^0(\mathcal{J}_{Z'}(H-A)) \geq 3$ and we claim that $A = 0$. Indeed, if $A \neq 0$, then the previous inequality tells us that A is a line, $\mathcal{J}_{Z'}(H-A) = \mathcal{O}_X(H-A)$, and $h^0(\mathcal{O}_X(H-A)) = 3$. From this, the exact sequence (7.14), and the inequality (7.13), we obtain

$$5 \leq h^0(\mathcal{O}_X(A)) + h^0(\mathcal{O}_X(H-A)) = h^0(\mathcal{O}_X(A)) + 3.$$

Equivalently, $h^0(\mathcal{O}_X(A)) \geq 2$, *i.e.*, a line on X moves in a linear system. But this is impossible on an irregular surface.

Once we know that $A = 0$, the exact sequence (7.14) becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{J}_{Z'}(H) \longrightarrow 0. \quad (7.15)$$

Together with (7.13) gives

$$h^0(\mathcal{J}_{Z'}(H)) \geq 4.$$

Hence either $Z' = p$, a single point, or $Z' = 0$. The first possibility is interpreted as p being a base point of $\mathcal{O}_X(K_X + H)$. Since $d = H^2 \geq 5$, we can apply [29, Theorem 1] to deduce that X must be a scroll. Furthermore, by the proof of Proposition 7.2, X is an elliptic scroll of degree $d = 5$. This together with (7.11) tells us that

$$25 = d^2 = c_2(\mathcal{N}_X) = \deg(Z) + 3d = \deg(Z) + 15$$

or, equivalently, that $\deg(Z) = 10$. However, from (7.15) and (7.12) it follows that

$$1 = \deg(Z') = c_2(\mathcal{E}(-H)) = \deg(Z) - 2H^2 = 10 - 2 \cdot 5 = 0$$

which is absurd.

We turn now to the second possibility, $Z' = 0$. The exact sequence (7.15) takes the form

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{O}_X(H) \longrightarrow 0$$

and this implies that $\mathcal{E}(-H) \cong \mathcal{O}_X(H) \oplus \mathcal{O}_X$ since $\text{Ext}^1(\mathcal{O}_X(H), \mathcal{O}_X) \cong H^1(\mathcal{O}_X(-H)) = 0$. Geometrically, this means that Z is a complete intersection of two effective divisors H_1 and H_2 in $|H|$ and $|2H|$ respectively. In particular, we obtain

$$H^0(\mathcal{J}_Z(2H)) = \{h_1h + \gamma h_2 \mid h \in H^0(\mathcal{O}_X(H)), \gamma \in \mathbb{C}\}, \quad (7.16)$$

where h_i is a section corresponding to H_i , for $i = 1, 2$. Now, from the proof of Lemma 6.2, we recall that $H^0(\mathcal{J}_Z(2H))$ contains the 5-dimensional subspace

$$H^0(\mathfrak{J}_{V_3}(2))|_X = \{\partial_v(f)|_X \mid v \in H^0(\mathcal{O}_X(H))^*\}$$

spanned by the restriction to X of the partial derivatives of f , a homogeneous polynomial defining V_3 . From this and the description of $H^0(\mathcal{J}_Z(2H))$ in (7.16) it follows that the intersection

$$H^0(\mathfrak{J}_{V_3}(2))|_X \cap h_1 H^0(\mathcal{O}_X(H))$$

is a 4-dimensional vector space. This implies that we can choose homogeneous coordinate functions X_i , $i = 0, \dots, 4$, in \mathbb{P}^4 so that the partial derivatives have the form

$$\frac{\partial f}{\partial X_i} = \tilde{h}_1 T_j \quad \text{for } 0 \leq j \leq 3$$

where the T_j 's are some linear forms and on $H^0(\mathcal{O}_X(H))^*$ and \tilde{h}_1 is the linear form corresponding to h_1 . It follows that the remaining partial derivative $\frac{\partial f}{\partial X_4}$ defines a quadric hypersurface Y which intersects the hyperplane ($\tilde{h}_1 = 0$) along a quadric surface Q contained in the singular locus of the cubic hypersurface V_3 . This means that the secant variety of Q is contained in V_3 . Since the latter is irreducible, it follows that Q is a double plane. But such a plane must intersect X along a 1-dimensional subscheme which is contrary to our assumption.

Case $h^0(\mathcal{O}_X(A)) \geq 3$. We go back to the destabilizing sequence (7.14). We argue as in the proof of Lemma 7.3 to deduce that $H^0(\mathcal{O}_X(A)) \hookrightarrow H^0(\mathcal{E}(-H))$ is a proper subspace. Hence the divisor $D = H - A$ is effective. This gives

$$3 \leq h^0(\mathcal{O}_X(A)) = h^0(\mathcal{O}_X(H - D)).$$

Thus D is either a line and the inequality above must be equality, or $D = 0$. The first possibility implies

$$5 \leq h^0(\mathcal{E}(-H)) \leq h^0(\mathcal{O}_X(A)) + h^0(\mathcal{J}_{Z'}(D)) = 3 + h^0(\mathcal{J}_{Z'}(D))$$

with the conclusion that the line D moves in a linear system on X contradicting the hypothesis X irregular. The second possibility, $D = 0$, implies $A = H$. In this case the destabilizing sequence (7.14) takes the form

$$0 \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since the epimorphism above induces a surjection on the level of global sections, we deduce again that $\mathcal{E}(-H) \cong \mathcal{O}_X(H) \oplus \mathcal{O}_X$, a situation discarded in the first part of the proof. \square

Next we investigate the possibility of V_3 being a cone over a cubic surface in \mathbb{P}^3 .

LEMMA 7.6. *If the cubic hypersurface V_3 is a cone, then it contains no smooth irregular surface.*

Proof. We begin with the case of V_3 being a cone with vertex at a point x_0 over a cubic surface S which is not a cone. We go back to the sequence (7.1) and adapt our arguments thereafter to the case at hand.

Claim. If $Z_1 \neq 0$, then $H^1(\mathcal{J}_{Z_0}(K_X + 3H - Z_1)) = 0$.

We proceed as in the proof of Lemma 7.3 by studying the extension sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z_0}(3H - Z_1) \longrightarrow 0.$$

As before we have the isomorphism $H^0(\mathcal{E}(-H)) \cong H^0(\mathcal{J}_{Z_0}(2H - Z_1))$ and from the proof of Lemma 6.2, it follows that

$$h^0(\mathcal{J}_{Z_0}(2H - Z_1)) \geq 4.$$

This inequality and the isomorphism just above it give

$$h^0(\mathcal{E}(-H)) \geq 4.$$

This leads again to the exact sequence (7.6)

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{J}_{Z'}(H - Z_1 - A) \longrightarrow 0,$$

where A is an effective divisor and $H^0(\mathcal{J}_{Z'}(H - Z_1 - A)) \neq 0$. Arguing as in the proof of Lemma 7.3 and using the assumption that the base of the cone V_3 is not a cone, we obtain

$$h^0(\mathcal{O}_X(A)) = h^0(\mathcal{J}_{Z'}(H - Z_1 - A)) = 2. \quad (7.17)$$

Set A_0 (resp. A') the fixed (resp. moving) part of $|A|$ and consider $\mathcal{O}_X(H - Z_1 - A_0)$. It is easy to see that $h^0(\mathcal{J}_{Z'}(H - Z_1 - A_0)) = 3$. From this and (7.17) it follows that $A_0 = 0$, Z_1 is a line, $Z' = 0$, and the linear map

$$H^0(\mathcal{O}_X(A)) \otimes H^0(\mathcal{O}_X(H - Z_1 - A)) \longrightarrow H^0(\mathcal{O}_X(H - Z_1))$$

has a nontrivial kernel. Hence there is a basis $\{x, x'\}$ of $H^0(\mathcal{O}_X(A))$ and nonzero elements $y, y' \in H^0(\mathcal{O}_X(H - Z_1 - A))$ such that

$$xy - x'y' = 0$$

in $H^0(\mathcal{O}_X(H - Z_1))$. Furthermore, $xy - x'y'$ can be viewed as the restriction of an element from $\text{Sym}^2 H^0(\mathcal{O}_X(H))$. Since no quadric hypersurface contains X the above equality implies that $y = \lambda x'$ and $y' = \lambda x$, for some nonzero $\lambda \in \mathbb{C}$. In particular, we obtain $\mathcal{O}_X(A) = \mathcal{O}_X(H - Z_1 - A)$ or, equivalently,

$$\mathcal{O}_X(H - Z_1) = \mathcal{O}_X(2A). \quad (7.18)$$

Since $h^0(\mathcal{O}_X(2A)) = h^0(\mathcal{O}_X(H - Z_1)) = 3$, we deduce $\text{Sym}^2 H^0(\mathcal{O}_X(A)) \cong H^0(\mathcal{O}_X(2A))$. Hence $|A|$ is base point free and therefore, $A^2 = 0$. This and (7.18) imply that the linear system $|2A| = |H - Z_1|$ is composed with a pencil, *i.e.*, the morphism defined by $|H - Z_1|$ factors as follows,

$$\varphi_{|H-Z_1|} : X \xrightarrow{|A|} \mathbb{P}^1 \xrightarrow{|\mathcal{O}_{\mathbb{P}^1}(2)|} \mathbb{P}^2.$$

Equivalently, the projection from the line Z_1 maps X onto a conic. But this means that X is contained in a quadric hypersurface, contrary to the hypothesis that 3 is the least degree of a hypersurface containing X . This completes the proof of the claim.

Next we investigate the cohomology group $H^1(\mathcal{O}_X(2K_X + 2H + Z_1))$ in the exact sequence (7.2). For this we need to control the divisor Z_1 . This divisor depends on the singularities of the cubic surface S , the base of the cone V_3 . The following two possibilities may occur:

- 1) S has isolated singular points and then Z_1 is composed of one or two rulings of the cone V_3 ;
- 2) the singular locus L of S is a line and then Z_1 is a divisor contained in the plane P spanned by L and the vertex x_0 of the cone V_3 .

The first possibility is dealt with in the same way as in the proof of Lemma 7.4, so we turn to the second possibility. A hyperplane V passing through P intersects V_3 along the decomposable surface

$$V \cdot V_3 = 2P + P_V,$$

where P_V is the residual plane. Hence the divisor $H_V = V \cdot X$ has the form

$$H_V = 2Z_1 + F_V,$$

where F_V is the divisor residual to $2Z_1$ and contained in the plane P_V . As in the proof of Lemma 7.4, we relate $H^1(\mathcal{O}_X(2K_X + 2H + Z_1))$ to $H^1(\mathcal{O}_X(2K_X + 2H))$ via the sequence

$$H^1(\mathcal{O}_X(2K_X + 2H)) \longrightarrow H^1(\mathcal{O}_X(2K_X + 2H + Z_1)) \longrightarrow H^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)) \quad (7.19)$$

and investigate the cohomology group on the right of this sequence.

Since the divisor $Z_1 = P \cdot X$ is the scheme-theoretic intersection of a plane with the surface X , a result of Ellia and Folegatti, [14], implies that Z_1 is a reduced divisor. Hence, for an irreducible component C of Z_1 , we have

$$h^1(\mathcal{O}_C(2K_X + 2H + Z_1)) = h^1(\omega_C \otimes \mathcal{O}_C(K_X + 2H + Z_1^C)) = h^0(\mathcal{O}_C(-(K_X + 2H) - Z_1^C),$$

where ω_C is the dualizing sheaf of C , Z_1^C the component of Z_1 complementary to C and the second equality is the Serre duality on C . Since $\mathcal{O}_X(K_X + 2H)$ is base point free, see [29], the above identity tells us that $h^1(\mathcal{O}_C(2K_X + 2H + Z_1)) = 0$ unless $Z_1 = C$ and $\mathcal{O}_C(K_X + 2H) = \mathcal{O}_C$. But then $H \cdot C = 1$ and $C^2 = 0$. Hence X is a scroll and by Lemma 7.8, it is an elliptic scroll of degree $d = 5$. Since this possibility was ruled out in the proof of Lemma 7.4, we obtain $H^1(\mathcal{O}_{Z_1}(2K_X + 2H + Z_1)) = 0$. This together with (7.19) imply that the nonvanishing of $H^1(\mathcal{O}_X(2K_X + 2H + Z_1))$ can only occur if $H^1(\mathcal{O}_X(2K_X + 2H)) \neq 0$. This, in view of Proposition 7.2, implies that X is an elliptic scroll of degree $d = 5$ and Z_1 is a ruling of X . Thus we are back in the situation ruled out in the proof of Lemma 7.4.

Our considerations are now reduced to the case when the divisorial part Z_1 in the exact sequence (7.1) is zero. Hence that sequence has the form

$$0 \longrightarrow \mathcal{O}_X(K_X + 2H) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{J}_{Z_0}(3H) \longrightarrow 0$$

and, as before, the irregularity of X is controlled by the groups $H^1(\mathcal{O}_X(2K_X + 2H))$ and $H^1(\mathcal{J}_{Z_0}(3H + K_X))$.

According to Proposition 7.2, the nonvanishing of $H^1(\mathcal{O}_X(2K_X + 2H))$ may occur only if X is an elliptic scroll of degree $d = 5$. This degree is a numerical obstacle for X to be contained in a cubic cone. Indeed, take a general hyperplane V_1 in \mathbb{P}^4 complementary to x_0 , the vertex of the cone V_3 , and consider the projection of X from x_0 into V_1 . This gives rise to a (rational) map

$$f_{x_0} : X \dashrightarrow V_1 \quad (7.20)$$

onto a cubic surface $S := V_1 \cap V_3$. The degree of this map is as follows:

$$\deg(f_{x_0}) = \begin{cases} \frac{d}{3} & \text{if } x_0 \notin X, \\ \frac{d-1}{3} & \text{if } x_0 \in X. \end{cases} \quad (7.21)$$

For $d = 5$ none of the above values is an integer. Hence the vanishing of $H^1(\mathcal{O}_X(2K_X + 2H))$.

Claim. $H^1(\mathcal{J}_{Z_0}(3H + K_X)) = 0$.

The proof of this assertion follows the same pattern as the one of Lemma 7.5. Namely, the nonvanishing of $H^1(\mathcal{J}_{Z_0}(3H + K_X))$ is interpreted as a nontrivial extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z_0}(3H) \longrightarrow 0. \quad (7.22)$$

This implies

$$h^0(\mathcal{E}(-H)) = h^0(\mathcal{J}_{Z_0}(2H)) \geq 4, \quad (7.23)$$

where the inequality comes from the assumption that the surface S , the base of the cone V_3 , is not a cone; see the proof of Lemma 6.2 for details. In particular, as in the proof of Lemma 7.5, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{J}_{Z'}(H - A) \longrightarrow 0, \quad (7.24)$$

where A is an effective divisor and $h^0(\mathcal{J}_{Z'}(H - A)) \geq 1$. Putting this sequence together with the sequence (7.22), we obtain the relation

$$\deg(Z_0) - 2d = \deg(Z') + A \cdot (H - A). \quad (7.25)$$

We wish to understand the geometric ingredients—the divisor A and the 0-dimensional subscheme Z' —involved in this relation.

To begin with, we observe that A is nonzero. Indeed, if $A = 0$, then the relation (7.25) reads

$$\deg(Z') = \deg(Z_0) - 2d. \quad (7.26)$$

To make use of this relation, as well as of (7.25) later on, we give an upper bound for $\deg(Z_0)$. Namely, from the fact that the divisorial part Z_1 is zero, it follows that the cubic surface S , the base of the cone V_3 , has only isolated singularities and that the map f_{x_0} in (7.20) is finite outside x_0 . Furthermore, the subscheme Z_0 is the scheme-theoretic intersection of X with the rulings of the cone V_3 over the singular locus $\text{Sing}(S)$ of S . Hence we get the upper bound

$$\deg(Z_0) \leq \deg(f_{x_0}) \deg(\text{Sing}(S)) \leq \frac{d}{3} \deg(\text{Sing}(S)),$$

where the last inequality comes from (7.21). In addition, from the classification of normal cubic surfaces that are not cones, see [12, p. 448], it follows that $\deg(\text{Sing}(S)) \leq 6$. Substituting this estimate into the above inequality, we obtain

$$\deg(Z_0) \leq 2d. \quad (7.27)$$

This and the relation (7.26) imply $\deg(Z') = 0$. Hence the exact sequence (7.24) takes the form

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{O}_X(H) \longrightarrow 0,$$

situation we have already considered in the proof of Lemma 7.5.

We know now that the divisor A in (7.24) must be nonzero. Assume $h^0(\mathcal{O}_X(A)) = 1$. From this and (7.23) it follows that

$$h^0(\mathcal{J}_{Z'}(H - A)) \geq 3$$

implying that A is a line, the above inequality is an equality, and $Z' = 0$. This and (7.25) imply

$$\deg(Z_0) - 2d = A \cdot (H - A) = 1 - A^2,$$

which, together with the upper bound (7.27), implies $A^2 \geq 1$. Hence X is regular (one sees easily that X is a rational surface). Thus we may assume $h^0(\mathcal{O}_X(A)) \geq 2$. In fact the equality must hold, since, if $h^0(\mathcal{O}_X(A)) \geq 3$, then the divisor $L = H - A$ is a line and the relation (7.25) reads

$$\deg(Z_0) - 2d = \deg(Z') + (H - L) \cdot L = \deg(Z') + 1 - L^2.$$

This and the upper bound (7.27) imply $L^2 \geq 1$, the situation we have already discarded.

From $h^0(\mathcal{O}_X(A)) = 2$ and the inequality (7.23) we deduce

$$h^0(\mathcal{J}_{Z'}(H - A)) \geq 2.$$

Furthermore, if the inequality is strict, we arrive again at the situation ruled out previously: A is a line moving in a linear system. Thus

$$h^0(\mathcal{O}_X(A)) = h^0(\mathcal{J}_{Z'}(H - A)) = 2,$$

a situation we have already encountered in the first part of the proof. Arguing as there, we show that the linear system $|A|$ has at most a 0-dimensional base locus. Hence $A \cdot (H - A) \geq 0$. This, the relation (7.25), and the upper bound (7.27) imply that all inequalities involved must be equalities. In particular,

$$A \cdot (H - A) = 0.$$

Since both A and $(H - A)$ are effective, nonzero divisors that add up to H , a very ample divisor, the above identity is impossible. This completes the proof of the lemma in the case the surface S , the base of the cone V_3 , is not a cone.

We now turn to the case when S is a cone with vertex x_S over a plane cubic curve C . If C is smooth, then V_3 is a singular scroll ruled by the planes $P_t = \text{Span}(t \cup L)$, for $t \in C$, with the singular locus $\text{Sing}(V_3)$ being the line $L = \text{Span}(x_0 \cup x_S)$. This is analogous to the situation we arrived at in the proof of Theorem 6.1. Hence the sequence (7.1) takes the form¹³

$$0 \longrightarrow \mathcal{O}_X(K_X + 2H + L) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{O}_X(3H - L) \longrightarrow 0.$$

It follows that the irregularity of X is controlled by the cohomology group $H^1(\mathcal{O}_X(2K_X + 2H + L))$. Its nonvanishing, as we have seen on several occasions, may occur only if X is an elliptic scroll of degree $d = 5$ and L is a ruling of the scroll. But computing the second Chern classes from the above sequence, we obtain

$$25 = d^2 = (K_X + 2H + L) \cdot (3H - L) = 18,$$

an obvious contradiction.

If C is singular, let c_0 be its (unique) singular point. The plane $P_{c_0} = \text{Span}(c_0 \cup L)$ is the singular plane of V_3 . Furthermore, the planes $P_t = \text{Span}(t \cup L)$ give rise to a rational family of curves

$$\{F_t = P_t \cdot X \mid t \in C\}.$$

Let F be the divisor class of this family. Then by taking a general hyperplane in \mathbb{P}^4 passing through the line L , we have

$$\mathcal{O}_X(H) = \mathcal{O}_X(3F).$$

From $h^0(\mathcal{O}_X(F)) = h^0(\mathcal{O}_X(H - 2F))$ it follows that $h^0(\mathcal{O}_X(F)) = 2$. Since the linear map

$$\text{Sym}^2(H^0(\mathcal{O}_X(F))) \longrightarrow H^0(\mathcal{O}_X(2F))$$

is injective, we deduce

$$3 \leq h^0(\mathcal{O}_X(2F)) = h^0(\mathcal{O}_X(H - F))$$

and hence F is a line. Since F moves in a pencil the surface X must be rational. This completes the proof of the lemma. \square

¹³We consider here only the case $L \subset X$, since the other case, $L \not\subset X$, is treated in exactly the same way as in the proof of Theorem 6.1.

We know now that an irregular X contained in a cubic hypersurface must be an elliptic scroll of degree 5. Furthermore, we have the following.

LEMMA 7.7. *Let X be an elliptic scroll of degree $d = 5$ in \mathbb{P}^4 and let V_3 be a cubic hypersurface containing it. Then the scheme $Z = X \cap \text{Sing}(V_3)$ is 0-dimensional of degree 10.*

Proof. From the proof of Lemma 7.6 we know that V_3 is not a cone. Combining this with Lemma 7.4, we deduce that the scheme $Z = X \cap \text{Sing}(V_3)$ is 0-dimensional. Hence the sequence (7.1) has the form

$$0 \longrightarrow \mathcal{O}_X(K_X + 2H) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{J}_Z(3H) \longrightarrow 0.$$

Computing the second Chern classes from this sequence, we obtain

$$\deg(Z) = \deg\left(X \cap \text{Sing}(V_3)\right) = 10.$$

□

To make the paper self-contained, we include the following result which is a well-known part of the classification of surfaces in \mathbb{P}^4 , see [22].

LEMMA 7.8. *The only irrational scrolls in \mathbb{P}^4 are elliptic scrolls of degree 5.*

Proof. Let X be a \mathbb{P}^1 -bundle over a smooth connected curve B of genus $q \geq 1$ and let $\mathcal{O}_X(H)$ be a very ample line bundle on X defining an embedding of X into \mathbb{P}^4 as a scroll. In particular, a smooth divisor in the linear system $|H|$ is isomorphic to B . This together with the adjunction formula implies

$$H \cdot K_X + H^2 = H \cdot K_X + d = 2q - 2.$$

Substituting this and the Chern numbers $\chi(\mathcal{O}_X) = 1 - q$, $K_X^2 = 8(1 - q)$ into the double point formula, we obtain

$$d^2 - 5d = 6(q - 1). \quad (7.28)$$

The main point of the argument is to compare the genus q of B in the above formula with the Castelnuovo upper bound on the genus of a smooth curve in a projective space. Namely, the scroll $X \subset \mathbb{P}^4$ is interpreted as an embedding

$$\varphi : B \longrightarrow \text{Gr}(1, \mathbb{P}^4)$$

into the Grassmannian $\text{Gr}(1, \mathbb{P}^4)$ of lines in \mathbb{P}^4 . Setting \mathcal{G} to be the pullback under φ of the universal subbundle of the Grassmannian, we identify X with the projectivization $\mathbb{P}(\mathcal{G})$. Then $\mathcal{O}_X(H)$, the line bundle embedding X into \mathbb{P}^4 , is such that the direct image $\pi_*(\mathcal{O}_X(H)) \cong \mathcal{G}^*$. In particular,

$$\deg(c_1(\mathcal{G}^*)) = d.$$

Composing φ with the Plücker embedding of $\text{Gr}(1, \mathbb{P}^4)$ gives the embedding

$$\psi : B \hookrightarrow \mathbb{P}^9$$

realized by the subsystem of $|\wedge^2 \mathcal{G}^*|$ corresponding to the image of the obvious homomorphism

$$\rho : \wedge^2 H^0(\mathcal{G}^*) \longrightarrow H^0(\wedge^2 \mathcal{G}^*).$$

We claim that $\ker(\rho)$ has dimension at least 5. Indeed, assume

$$\dim(\ker(\rho)) \leq 4.$$

Then the image of ρ has dimension $N + 1 \geq 6$ and ψ embeds B into \mathbb{P}^N as a nondegenerate curve of degree d . The Castelnuovo bound on the genus q in \mathbb{P}^N gives

$$q - 1 \leq sd - \frac{s(s+1)(N-1)}{2} - N - 1,$$

where $d = s(N-1) + r$ for an integer $0 \leq r < N-1$. Rewriting the expression on the right as a function of d , r , and N , we obtain

$$q - 1 \leq \frac{1}{2(N-1)} (d^2 - r^2) - \frac{1}{2}d + \frac{r}{2} - (N+1),$$

In our situation $N \in \{5, \dots, 9\}$, hence we can consider the larger bound

$$q - 1 \leq \frac{1}{8}d^2 - \frac{1}{2}d - \frac{r^2}{16} + \frac{r}{2} - 6.$$

Putting it together with (7.28) gives

$$d^2 - 5d \leq \frac{3}{4}d^2 - 3d - \frac{3r^2}{8} + 3r - 36.$$

This can be rewritten in the form

$$\left(\frac{1}{2}d - 2\right)^2 = \frac{1}{4}d^2 - 2d + 4 \leq -\frac{3r^2}{8} + 3r - 32 \leq -26$$

which is false.

We know now that the kernel of ρ is at least of dimension 5. Geometrically this means that the projectivized subspace $\mathbb{P}(\ker(\rho))$ intersects the Grassmannian variety $\text{Gr}(2, H^0(\mathcal{G})) \cong \text{Gr}(1, \mathbb{P}^4)$ of decomposable tensors in $\mathbb{P}(\wedge^2 H^0(\mathcal{G}^*)) \cong \mathbb{P}^9$ along a subscheme of dimension at least 1. Each decomposable tensor $g \wedge g'$ in this intersection, viewed as a section of $\wedge^2 \mathcal{G}^*$, is zero. Equivalently, the two sections g and g' correspond, under the isomorphism $H^0(\mathcal{G}^*) \cong H^0(\mathcal{O}_X(H))$, to two hyperplanes H_g and $H_{g'}$ in \mathbb{P}^4 such that the plane $P = H_g \cap H_{g'}$ intersects the scroll X along a curve $\Gamma_{g,g'}$ which is a section of the structure projection $\pi : X \rightarrow B$. The above shows that there is a family $\{\Gamma_t\}_{t \in T}$ of such sections parametrized by an irreducible curve T .

Observe that every two planes P_t and $P_{t'}$, for $t \neq t'$, intersect at a single point. This implies that $\Gamma_t \cdot \Gamma_{t'} \leq 1$. We easily check that $\Gamma_t \cdot \Gamma_{t'} = 1$. In particular, setting Γ to be the class of $\{\Gamma_t\}_{t \in T}$ in the Néron-Severi group of X , we obtain

$$\Gamma^2 = 1.$$

We write $H = \Gamma + aF$, where F is the class of a ruling of X . Then $d = H^2 = 2a + 1$ and the degree of Γ is $d_\Gamma = H \cdot \Gamma = d - a = 2a + 1 - a = a + 1$. Since Γ is a plane curve, we have

$$2(q-1) = (a+1)(a-2).$$

This, the identity $d = 2a + 1$ and (7.28) imply

$$3(a+1)(a-2) = 6(q-1) = d^2 - 5d = (2a+1)^2 - 5(2a+1) = (2a+1)(2a-4)$$

or, equivalently,

$$(a-2)(a-1) = 0.$$

This leads to two solutions $d = 5$ and 3 , which are, respectively, an elliptic scroll of degree 5 and a rational scroll of degree 3. \square

8. ON THE ELLIPTIC SCROLL OF DEGREE 5 AND THE SEGRE CUBIC

In the previous section we have characterized an elliptic scroll X of degree 5 in \mathbb{P}^4 as being the only irregular surface lying on a cubic hypersurface. Such scrolls are notorious and they have been subject to extensive study; see, *e.g.*, [18, 4, 5] and the references therein. The main objective of this section is to (re)establish a relationship between two entities related to the embedding of X in \mathbb{P}^4 :

- the space $I_X(3)$ of cubic hypersurfaces in \mathbb{P}^4 containing X
- the space of global sections $H^0(\mathcal{N}_X^*(3H))$ of the conormal bundle \mathcal{N}_X^* of X in \mathbb{P}^4 twisted by $\mathcal{O}_X(3H)$, where $\mathcal{O}_X(H)$ is a line bundle realizing the embedding of X in \mathbb{P}^4 .

Using the above notation, we formulate the main result of this section.

THEOREM 8.1. *Let X be an elliptic scroll of degree 5.*

- 1) *The sheaf $\mathcal{N}_X^*(3H)$ is a rank 2 vector bundle generated by its global sections, with Chern invariants $c_1(\mathcal{N}_X^*(3H)) = H - K_X$ and $c_2(\mathcal{N}_X^*(3H)) = 10$.*
- 2) *There is a natural isomorphism $H^0(\mathcal{N}_X^*(3H)) \cong I_X(3) \cong \mathbb{C}^5$.*
- 3) *Every nonzero global section s of $\mathcal{N}_X^*(3H)$ has 0-dimensional zero locus of degree 10. Under the above correspondence, the scheme of zeros $Z_s = (s = 0)$ is the scheme-theoretic intersection of X with $\text{Sing}(V_3(s))$, the singular locus of the cubic hypersurface $V_3(s) \in |I_X(3)|$ corresponding to s . In particular, every global section s with $Z_s = (s = 0)$ consisting of ten distinct points, corresponds to a Segre cubic $V_3(s)$, whose set of nodes $\text{Sing}(V_3(s)) = Z_s$.*

Proof. The assertion about the Chern invariants is obvious. Of course the relation between global sections of $\mathcal{N}_X^*(3H)$ and cubic hypersurfaces through X has been at the origin of our considerations in Section 7 and stems from the identification $\mathcal{N}_X^* = \mathcal{I}_X / \mathcal{I}_X^2$, where \mathcal{I}_X is the ideal sheaf of X in \mathbb{P}^4 . From the exact sequence

$$0 \longrightarrow \mathcal{I}_X^2(3) \longrightarrow \mathcal{I}_X(3) \longrightarrow \mathcal{N}_X^*(3H) \longrightarrow 0$$

follows the inclusion¹⁴

$$0 \longrightarrow I_X(3) = H^0(\mathcal{I}_X(3)) \longrightarrow H^0(\mathcal{N}_X^*(3H)). \quad (8.1)$$

From the exact sequence $0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow \mathcal{O}_X \rightarrow 0$ tensored with $\mathcal{O}_{\mathbb{P}^4}(3)$ we obtain the estimate

$$h^0(\mathcal{I}_X(3)) \geq h^0(\mathcal{O}_{\mathbb{P}^4}(3)) - h^0(\mathcal{O}_X(3H)) = \binom{7}{3} - \frac{1}{2}(9H^2 - 3H \cdot K_X) = 35 - 30 = 5, \quad (8.2)$$

where we used the easily verified property

$$H^i(\mathcal{O}_X(kH)) = 0 \quad \text{for all } k > 0 \quad \text{and } i = 1, 2.$$

The above estimate is actually an equality.¹⁵ This as well as the isomorphism in (8.1), and hence the assertion 2) of the theorem, follow from the next lemma.

¹⁴The inclusion comes from $H^0(\mathcal{I}_X^2(3)) = 0$, since we know that X is not contained in a quadric hypersurface, see Theorem 6.1.

¹⁵One easily verifies that $X \subset \mathbb{P}^4$ is a projectively normal embedding. Since we do not use this aspect anywhere, the above mentioned equality is established differently in the proof of Lemma 8.2.

LEMMA 8.2. *The vector bundle $\mathcal{N}_X^*(3H)$ is globally generated and $h^0(\mathcal{N}_X^*(3H)) = 5$.*

Proof. Let $\pi : X \rightarrow E$ be the structure morphism, i.e., E is an elliptic curve and π is a \mathbb{P}^1 -fibration over E . The line bundle $\mathcal{O}_X(H)$ defining the embedding of X into \mathbb{P}^4 as a scroll has degree 1 on the fibres of π , i.e.,

$$\mathcal{O}_X(H) \otimes \mathcal{O}_F = \mathcal{O}_F(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$$

on every fibre F of π .

We wish to understand the restriction of $\mathcal{N}_X^*(3H)$ to a fibre F . Since $\det(\mathcal{N}_X) = \mathcal{O}_X(5H + K_X)$, we have

$$\det(\mathcal{N}_X) \otimes \mathcal{O}_F = \mathcal{O}_F(5H + K_X) = \mathcal{O}_F(3).$$

From this it follows that

$$\mathcal{N}_X^*(3H) \otimes \mathcal{O}_F = \mathcal{N}_X^* \otimes \mathcal{O}_F(3) = \mathcal{N}_X^* \otimes \det(\mathcal{N}_X) \otimes \mathcal{O}_F \cong \mathcal{N}_X \otimes \mathcal{O}_F \cong \mathcal{O}_F(1) \oplus \mathcal{O}_F(2), \quad (8.3)$$

where the last isomorphism follows from the global generation of $\mathcal{N}_X(-H) \otimes \mathcal{O}_F = \mathcal{N}_X \otimes \mathcal{O}_F(-1)$. In particular, the restriction $\mathcal{N}_X^*(3H) \otimes \mathcal{O}_F$ is globally generated on every fibre F of π . So to obtain the global generation of $\mathcal{N}_X^*(3H)$, it is sufficient to show the surjectivity

$$H^0(\mathcal{N}_X^*(3H)) \longrightarrow H^0(\mathcal{N}_X^*(3H) \otimes \mathcal{O}_F) = H^0(\mathcal{O}_F(1) \oplus \mathcal{O}_F(2))$$

for every fibre F of π . For this we use the inclusion (8.1) to obtain the composition

$$\begin{array}{c} 0 \\ \downarrow \\ I_X(3) \\ \downarrow \quad \searrow \\ 0 \longrightarrow H^0(\mathcal{N}_X^*(3H - F)) \longrightarrow H^0(\mathcal{N}_X^*(3H)) \longrightarrow H^0(\mathcal{N}_X^*(3H) \otimes \mathcal{O}_F) \longrightarrow 0 \end{array} \quad (8.4)$$

In addition, from the proof of Theorem 7.1, we know that the sections of $\mathcal{N}_X^*(3H)$ coming from $I_X(3)$ have 0-dimensional schemes of zeros. Hence, the image of $I_X(3)$ in $H^0(\mathcal{N}_X^*(3H))$ is complementary to the kernel of the restriction homomorphism

$$H^0(\mathcal{N}_X^*(3H)) \longrightarrow H^0(\mathcal{N}_X^*(3H) \otimes \mathcal{O}_F)$$

in (8.4). From this it follows that the slanted arrow in (8.4) is injective. Since the target of that arrow is a space of dimension 5, see (8.3), we deduce the inequality $\dim(I_X(3)) \leq 5$. This and the estimate (8.2) imply

$$\dim(I_X(3)) = 5. \quad (8.5)$$

Therefore, the slanted arrow in (8.4) is an isomorphism, hence the global generation of $\mathcal{N}_X^*(3H)$.

We now turn to the assertion $h^0(\mathcal{N}_X^*(3H)) = 5$. From (8.4), we already know that $h^0(\mathcal{N}_X^*(3H)) \geq 5$. Let us assume that the inequality is strict. From the Koszul sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{N}_X^*(3H) \xrightarrow{\wedge^s} \mathcal{J}_{Z_s}(H - K_X) \longrightarrow 0 \quad (8.6)$$

of a general global section s of $\mathcal{N}_X^*(3H)$, we have

$$h^0(\mathcal{J}_{Z_s}(H - K_X)) \geq h^0(\mathcal{N}_X^*(3H)) - 1 \geq 5. \quad (8.7)$$

Furthermore, considering another general global section of $\mathcal{N}_X^*(3H)$, we obtain the smooth curve $\Gamma = (\gamma = 0)$, where γ is the section of $\det(\mathcal{N}_X^*(3H)) = \mathcal{O}_X(H - K_X)$ corresponding to $s \wedge s'$ under the natural homomorphism

$$\wedge^2 H^0(\mathcal{N}_X^*(3H)) \longrightarrow H^0(\det(\mathcal{N}_X^*(3H))) = H^0(\mathcal{O}_X(H - K_X)).$$

This gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\gamma} \mathcal{J}_{Z_s}(H - K_X) \longrightarrow \mathcal{O}_\Gamma((H - K_X)|_\Gamma - Z_s) \longrightarrow 0.$$

From this and (8.7) we obtain

$$h^0(\mathcal{O}_\Gamma((H - K_X)|_\Gamma - Z_s)) \geq h^0(\mathcal{J}_{Z_s}(H - K_X)) - 1 \geq 5 - 1 = 4. \quad (8.8)$$

On the other hand, the degree of $\mathcal{O}_\Gamma((H - K_X)|_\Gamma - Z_s)$ is

$$(H - K_X) \cdot \Gamma - \deg(Z_s) = (H - K_X)^2 - c_2(\mathcal{N}_X^*(3H)) = 15 - 10 = 5,$$

while the genus of Γ , by the adjunction formula, is 6. This implies that $\mathcal{O}_\Gamma((H - K_X)|_\Gamma - Z_s)$ is special and by the Clifford inequality

$$h^0(\mathcal{O}_\Gamma((H - K_X)|_\Gamma - Z_s)) \leq \frac{\deg((H - K_X)|_\Gamma - Z_s)}{2} + 1 = \frac{5}{2} + 1.$$

Combining this inequality with (8.8), we obtain

$$4 \leq h^0(\mathcal{O}_\Gamma((H - K_X)|_\Gamma - Z_s)) \leq \frac{5}{2} + 1,$$

an obvious contradiction. \square

End of the proof of Theorem 8.1. Since the spaces $H^0(\mathcal{N}_X^*(3H))$ and $I_X(3)$ are both 5-dimensional, see (8.5) for the latter, an immediate corollary of Lemma 8.2 is that the inclusion (8.1) is an isomorphism.

Once the isomorphism in Theorem 8.1, 2) is established, we also deduce that every nonzero global section of $\mathcal{N}_X^*(3H)$ has 0-dimensional scheme of zeros, since this is now equivalent to the property of cubics in $I_X(3)$ to produce sections of $\mathcal{N}_X^*(3H)$ with 0-dimensional zero locus, see Lemma 7.7. This proves the first assertion of Theorem 8.1, 3).

The degree of the scheme of zeros of a nonzero section of $\mathcal{N}_X^*(3H)$ is the value of the second Chern class (identified with its degree) $c_2(\mathcal{N}_X^*(3H))$ and this value is 10 by the part 1) of the theorem.

Next we turn to the assertion that Z_s , the zero scheme of a nonzero global section s of $\mathcal{N}_X^*(3H)$, is the scheme theoretic intersection of X with the singular locus of the cubic hypersurface $V_3(s)$ corresponding to s under the isomorphism in part 2) of the theorem. From the general discussion about the relation of the ideal sheaf \mathcal{J}_{Z_s} of Z_s and the restriction of the Jacobian ideal $\mathfrak{J}_{V_3(s)}$ to X we know that

$$\mathcal{J}_{Z_s} \supset \mathfrak{J}_{V_3(s)} \otimes \mathcal{O}_X,$$

i.e., Z_s is contained in the scheme theoretic intersection of X with the singular locus of the cubic hypersurface $V_3(s)$. To see the equality, it is enough to show that the generators of $\mathfrak{I}_{V_3(s)}$, the partial derivatives of a homogeneous polynomial defining $V_3(s)$, restricted to X , generate the sheaf $\mathcal{J}_{Z_s}(2H)$. This follows immediately from the epimorphism

$$\mathcal{N}_X(-H) \longrightarrow \mathcal{J}_{Z_s}(2H)$$

coming from the Koszul sequence (8.6) tensored with $\mathcal{O}_X(H + K_X)$ and the surjective morphism

$$H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X \longrightarrow \mathcal{N}_X(-H).$$

The resulting composition

$$H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X \longrightarrow \mathcal{J}_{Z_s}(2H)$$

is surjective and is described explicitly by the partial derivatives of a homogeneous polynomial defining $V_3(s)$, see the proof of Lemma 6.2 for details. Hence the asserted equality

$$\mathcal{J}_{Z_s} = \mathfrak{I}_{V_3(s)} \otimes \mathcal{O}_X. \quad (8.9)$$

We are left with the last assertion of 3) of the theorem, stating that sections of $\mathcal{N}_X^*(3H)$ with simple zeros correspond to Segre cubics in $|I_X(3)|$. Indeed, let s be a global section of $\mathcal{N}_X^*(3H)$ with $Z_s = (s = 0)$ consisting of ten distinct points. From the equality (8.9), we deduce that the singular locus $\text{Sing}(V_3(s))$ of the cubic $V_3(s)$ contains ten distinct points. It will be enough to show that $\text{Sing}(V_3(s))$ is 0-dimensional, since then (8.9) tells us that the singular locus $\text{Sing}(V_3(s)) = Z_s$ and it is composed of ten ordinary double points. It is well known that such a cubic hypersurface is a Segre cubic (see [11] for an inspiring introduction to the subject).

Let us check now that $\text{Sing}(V_3(s))$ is 0-dimensional. By Lemma 7.6, $V_3(s)$ is not a cone. Then the possibilities for the one dimensional part of $\text{Sing}(V_3(s))$ are a line, a conic (possibly singular), or a rational normal curve of degree 4 in \mathbb{P}^4 .

If a conic C is a component of $\text{Sing}(V_3(s))$, then the plane P spanned by C is contained in V_3 . We examine the pencil of hyperplanes V_t in \mathbb{P}^4 passing through P . The intersection $V_t \cdot V_3(s)$ is reducible

$$V_t \cdot V_3(s) = P \cup Q_t, \quad \forall t,$$

where Q_t is a quadric surface residual to the plane P such that $Q_t \cap P = C$. The hyperplane section

$$H_t = V_t \cdot X = B + \Gamma_t \quad (8.10)$$

is also reducible, where $B = P \cdot X$ is the 1-dimensional part of the base locus of the pencil $\{H_t\}$. Being a plane divisor, B can be either a ruling of X or its plane cubic section. The latter case implies $B \cdot C = 6$. So the part Z_C of the scheme Z_s contained in the intersection $B \cap C$ has degree at least 6. On the other hand the degree of the subscheme Z_B of Z_s contained in B is at most $B \cdot (H - K_X) = 4$ (we use here the fact that Z_s is contained in an irreducible divisor in the linear system $|H - K_X|$). Hence B must be a ruling of X . From this and (8.10) it follows that for a general t , the curve Γ_t is an elliptic curve of degree 4 contained in Q_t . In particular, the scheme-theoretic intersection $C \cap \Gamma_t$ is a 0-dimensional scheme of degree 4. Furthermore, since C is not contained in X , this scheme must be contained in the base locus of the pencil $\{\Gamma_t\}$. Thus we obtain

$$4 \leq \Gamma_t^2 = (H - B)^2 = 3$$

a contradiction.

If a rational normal curve R of degree 4 is in $\text{Sing}(V_3(s))$, then $\text{Sing}(V_3(s)) = R$ and $Z_s = X \cdot R$. It follows that

$$h^0(\mathcal{J}_{Z_s}(2H)) \geq h^0(\mathcal{J}_{R/\mathbb{P}^4}(2)) = 6, \quad (8.11)$$

where $\mathcal{J}_{R/\mathbb{P}^4}$ is the ideal sheaf of R in \mathbb{P}^4 . Since Z_s lies on a smooth curve $\Gamma \in |H - K_X|$, we can calculate $h^0(\mathcal{J}_{Z_s}(2H))$ from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-\Gamma) \longrightarrow \mathcal{J}_{Z_s} \longrightarrow \mathcal{O}_\Gamma(-Z_s) \longrightarrow 0$$

tensored with $\mathcal{O}_X(2H)$. This gives

$$h^0(\mathcal{J}_{Z_s}(2H)) = h^0(\mathcal{O}_\Gamma(2H|_\Gamma - Z_s)) = 5 + h^1(\mathcal{O}_\Gamma(2H|_\Gamma - Z_s)) = 5 + h^0(\mathcal{O}_\Gamma(Z_s - H|_\Gamma)).$$

This and (8.11) imply $h^0(\mathcal{O}_\Gamma(Z_s - H|_\Gamma)) = 1$ or, equivalently, $\mathcal{O}_\Gamma(Z_s) = \mathcal{O}_\Gamma(H)$ and this contradicts Corollary 8.5 below.

We turn now to the remaining case: the 1-dimensional locus of $\text{Sing}(V_3(s))$ is a line L . We divide the scheme Z_s into two parts,

$$Z_s = Z_L + Z',$$

where Z_L is the part of Z_s contained in L and $Z' = Z_s \setminus Z_L$ is the residual part. It is easy to see that $\deg(Z_L) \leq 3$. But then Z' consists of at least 7 distinct points. For every point $z' \in Z'$ the plane $P_{z'} = \text{Span}(z' \cup L)$ is contained in $V_3(s)$. Furthermore, $P_{z'_1} \neq P_{z'_2}$ for any pair of distinct points $z'_1, z'_2 \in Z'$ since otherwise the line $L' = \text{Span}\{z'_1, z'_2\}$ intersects L and hence, is a component of the singular locus of $V_3(s)$.

Every plane $P_{z'}$ intersects X along a plane curve. This curve is either a ruling of X or its plane cubic section. Assume there is $z'_0 \in Z'$ such that that $P_{z'_0} \cdot X = \Gamma$ is a plane cubic section. Then $\Gamma \cap L = Z_L$ and all other planes $P_{z'}, z' \neq z'_0$, intersect X along a ruling. Since those rulings must pass through one of the points of Z_L , the number of such planes is at most 3. This makes the degree of Z' at most 4. This is contrary to the estimate $\deg(Z') \geq 7$. Hence every plane $P_{z'}$ intersects X along a ruling. But then $3 \geq \deg(Z_L) \geq \deg(Z') \geq 7$ which is impossible. This completes the proof of the theorem. \square

Remark 8.3. From the proof of Theorem 8.1, 3), it follows that if a cubic hypersurface $V_3(s)$ contains X and has 1-dimensional singular locus, then its 1-dimensional part must be a single line. Furthermore, if this possibility occurs, the global section s of $\mathcal{N}^*(3H)$ corresponding to $V_3(s)$ under the isomorphism in Theorem 8.1, 2), must have multiple zeros. In the appendix, see (A.17) and the discussion preceding it, we give an explicit geometric construction of a general cubic in $|I_X(3)|$, singular along a line. Hence, that line is precisely the 1-dimensional part of the singular locus of such a cubic. In addition, we show that the isomorphism in Theorem 8.1, 2), matches precisely the global sections of $\mathcal{N}^*(3H)$ having multiple zeros with the cubics in $|I_X(3)|$ having a (unique) line in their singular locus, see Proposition A.10.

In the course of the proof of Theorem 8.1, 3), we used the fact that on a smooth curve $\Gamma \in |H - K_X|$ containing $Z_s = (s = 0)$, the zero scheme of a nonzero global section s of $\mathcal{N}^*(3H)$, the line bundles $\mathcal{O}_\Gamma(Z_s)$ and $\mathcal{O}_\Gamma(H)$ are not isomorphic; see Corollary 8.5. This is a part of the proof of the following.

LEMMA 8.4. *For every nonzero global section $s \in H^0(\mathcal{N}_X^*(3H))$ with $Z_s = (s = 0)$ one has $h^0(\mathcal{J}_{Z_s}(2H)) = 5$ and $h^1(\mathcal{J}_{Z_s}(2H)) = 0$.*

Proof. By Theorem 8.1, 3), the subscheme $Z_s = (s = 0)$ is 0-dimensional. Hence the Koszul sequence of s

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{N}_X^*(3H) \xrightarrow{s^\wedge} \mathcal{J}_{Z_s}(H - K_X) \longrightarrow 0$$

is exact. Tensoring it with $\mathcal{O}_X(K_X + H)$ and using the identification $\mathcal{N}_X^*(K_X + 4H) \cong \mathcal{N}_X(-H)$ we obtain

$$0 \longrightarrow \mathcal{O}_X(K_X + H) \xrightarrow{s} \mathcal{N}_X(-H) \xrightarrow{s^\wedge} \mathcal{J}_{Z_s}(2H) \longrightarrow 0.$$

From this it follows that $h^0(\mathcal{J}_{Z_s}(2H)) = h^0(\mathcal{N}_X(-H))$ is independent of the choice of s . So to compute $h^0(\mathcal{J}_{Z_s}(2H))$ we choose s with simple zeros and $s' \in H^0(\mathcal{N}_X^*(3H))$ so that the curve $\Gamma = (s \wedge s' = 0) \in |H - K_X|$ is smooth. The curve Γ passes through Z_s and gives the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-\Gamma) \xrightarrow{\gamma} \mathcal{J}_{Z_s} \longrightarrow \mathcal{O}_\Gamma(-Z_s) \longrightarrow 0, \quad (8.12)$$

where γ is the global section of $\mathcal{O}_X(H - K_X)$ corresponding to $s \wedge s'$ under the natural homomorphism $\wedge^2 H^0(\mathcal{N}_X^*(3H)) \rightarrow H^0(\det(\mathcal{N}_X^*(3H))) = H^0(\mathcal{O}_X(H - K_X))$. Tensoring the above sequence with $\mathcal{O}_X(2H)$, we deduce

$$h^0(\mathcal{J}_{Z_s}(2H)) = h^0(\mathcal{O}_\Gamma(2H|_\Gamma - Z_s)) = 5 + h^1(\mathcal{O}_\Gamma(2H|_\Gamma - Z_s)) = 5 + h^0(\mathcal{O}_\Gamma(Z_s - H|_\Gamma)),$$

where the second equality is the Riemann-Roch for $\mathcal{O}_\Gamma(2H|_\Gamma - Z_s)$ and the third one is the Serre duality. Thus the first assertion of the lemma is equivalent to

$$\mathcal{O}_\Gamma(Z_s) \neq \mathcal{O}_\Gamma(H). \quad (8.13)$$

Assume the contrary. Then the exact sequence (8.12) tensored with $\mathcal{O}_X(H - K_X)$ takes the form

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\gamma} \mathcal{J}_{Z_s}(H - K_X) \longrightarrow \mathcal{O}_\Gamma(-K_X) \longrightarrow 0.$$

This implies

$$h^0(\mathcal{O}_\Gamma(-K_X)) \geq h^0(\mathcal{J}_{Z_s}(H - K_X)) - 1 \geq h^0(\mathcal{N}_X^*(3H)) - 2 = 3. \quad (8.14)$$

On the other hand we have

$$0 \longrightarrow \mathcal{O}_X(-K_X - \Gamma) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_\Gamma(-K_X) \longrightarrow 0.$$

Since $\mathcal{O}_X(-K_X - \Gamma) = \mathcal{O}_X(-H)$, the above implies $H^0(\mathcal{O}_\Gamma(-K_X)) = H^0(\mathcal{O}_X(-K_X))$. However, we know that the last space is 1-dimensional. Hence $h^0(\mathcal{O}_\Gamma(-K_X)) = 1$ contradicting the estimate in (8.14).

The second assertion of the lemma about the vanishing of $H^1(\mathcal{J}_{Z_s}(2H))$ follows immediately from the first assertion and the Riemann-Roch for $\mathcal{J}_{Z_s}(2H)$. \square

COROLLARY 8.5. *Let s be a nonzero global section of $\mathcal{N}_X^*(3H)$ whose zero locus $Z_s = (s = 0)$ is contained in a smooth curve $\Gamma \in |H - K_X|$. Then the line bundle $\mathcal{O}_\Gamma(H|_\Gamma - Z_s) \neq \mathcal{O}_\Gamma$.*

Proof. The assertion is a restatement of the identity (8.13) proved in the previous lemma. \square

9. IRREGULAR SURFACES ON HYPERSURFACES OF DEGREE 4 WITH NON-DEGENERATE ISOLATED SINGULARITIES

In this section we consider irregular surfaces $X \subset \mathbb{P}^4$ contained in a hypersurface of degree 4 and not in one of a smaller degree. Our main result is as follows.

THEOREM 9.1. *Let $X \subset \mathbb{P}^4$ be a smooth surface with $m_X = 4$ and assume X to be contained in a quartic hypersurface V_4 with at most ordinary double points. Then X is regular, with the possible exception of X being a degree 8 elliptic conic bundle with $H \cdot K_X = 0$ and $K_X^2 = -8$. If such a situation occurs, then X must pass through precisely 32 singular points of V_4 .*

The exceptional possibility in the above theorem is an elliptic conic bundle discovered about 20 years ago by Abo, Decker, and Sasakura by using a certain vector bundle of rank 5 on \mathbb{P}^4 , see [1]. Shortly afterwards, Ranestad in [28], gave a geometric construction of a general elliptic conic bundle in \mathbb{P}^4 as the image of an elliptic scroll in \mathbb{P}^4 under a certain Cremona transformation. In the sequel we refer to those elliptic conic bundles as ADSR elliptic conic bundle.

From the first work cited above one knows that the space of quartics $H^0(\mathcal{J}_X(4))$ containing X is of dimension 6. However, at the time of writing this paper, we do not know if there are quartics in $H^0(\mathcal{J}_X(4))$ with only ordinary double points.

Our proof of Theorem 9.1 follows the same line of thinking as in the case of surfaces contained in hypersurfaces of degree 3. Namely, we assume $X \subset \mathbb{P}^4$ to be an irregular surfaces with $m_X = 4$ and lying on a quartic hypersurface V_4 with only ordinary double points. Our general situation recorded by the sequence (5.1) takes the form

$$0 \longrightarrow \mathcal{O}_X(K_X + H) \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{J}_Z(4H) \longrightarrow 0, \quad (9.1)$$

where Z is the 0-dimensional subscheme of X supported on the singular locus of V_4 and defined, at each point p of the support of Z , by the restriction to X of the Jacobian ideal $\mathfrak{J}_{V_4,p}$. In particular, we have¹⁶

$$\mathcal{J}_Z = \mathfrak{J}_{V_4} \otimes \mathcal{O}_X, \quad (9.2)$$

where \mathfrak{J}_{V_4} denotes the sheaf of the Jacobian ideal of V_4 .

Expressing the second Chern class of \mathcal{N}_X from the exact sequence (9.1) provides a new “double point formula”

$$d^2 = \deg(Z) + 4H \cdot (H + K_X) = \deg(Z) + 8(g - 1), \quad (9.3)$$

where $g = g(H)$ is the geometric genus of a general hyperplane section. In the sequel we refer to this identity as the ndp formula.

The cohomological sequence (5.3) controlling the irregularity of X becomes

$$H^1(\mathcal{O}_X(2K_X + H)) \longrightarrow \mathcal{N}_X(K_X) \longrightarrow H^1(\mathcal{J}_Z(K_X + 4H)). \quad (9.4)$$

Therefore, an understanding of the irregularity is reduced to the study of the cohomology groups $H^1(\mathcal{O}_X(2K_X + H))$ and $H^1(\mathcal{J}_Z(4H + K_X))$. In particular, we need to control the

¹⁶The identity (9.2) is valid as long as a section defining the Koszul sequence (9.1) has a 0-dimensional scheme of zeros.

scheme Z which, in view of the identity (9.2), comes down to controlling the singular locus $\text{Sing}(V_4)$ of V_4 . Under our assumption on the isolated singularities of V_4 , the singular locus $\text{Sing}(V_4)$ is the set of ordinary double points of V_4 and we can quote a result of A. Varchenko, [35], for the estimate $\deg(\text{Sing}(V_4)) \leq 45$. This together with (9.2) gives

$$\deg(Z) \leq \deg(\text{Sing}(V_4)) \leq 45. \quad (9.5)$$

With this estimate of $\deg(Z)$ recorded, we turn now to the study of the cohomology groups $H^1(\mathcal{O}_X(2K_X + H))$ and $H^1(\mathcal{J}_Z(K_X + 4H))$ in (9.4).

9.1. The study of $H^1(\mathcal{O}_X(2K_X + H))$

By the Serre duality, $H^1(\mathcal{O}_X(2K_X + H))^* = H^1(\mathcal{O}_X(-(K_X + H)))$. Thus the question of the (non)vanishing of this group comes down to understanding the geometric properties of the divisor $K_X + H$. The next lemma is an easy consequence of [29].

LEMMA 9.2. *Let X be an irregular surface and H be a very ample divisor on X . Then the following assertions hold.*

- 1) $\mathcal{O}_X(K_X + H)$ has base points if and only if X is a ruled surface and its embedding by $\mathcal{O}_X(H)$ is a scroll.
- 2) If $\mathcal{O}_X(K_X + H)$ is base point free and $H^1(\mathcal{O}_X(-(K_X + H))) \neq 0$, then X is a birationally ruled surface embedded by $\mathcal{O}_X(H)$ as a conic bundle over a smooth curve B of genus $q = q(X)$.

Proof. The assumption that X is irregular implies $H^2 \geq 5$. Then, by [29, Theorem 1, (i)], a base point of $\mathcal{O}_X(K_X + H)$ gives rise to an effective divisor $D \subset X$ passing through a base point of $|K_X + H|$ such that $H \cdot D = 1$ and $D^2 = 0$. It follows that D is a line in the embedding given by $\mathcal{O}_X(H)$. Hence the Albanese map $a : X \rightarrow \text{Alb}(X)$ must contract D to a point. The fact that $D^2 = 0$ implies that the map a factors through a smooth curve $B \subset \text{Alb}(X)$ of genus $q = q(X)$ and $a : X \rightarrow B$ is a \mathbb{P}^1 -fibration, with D one of the fibres. Thus X is a ruled surface embedded by $\mathcal{O}_X(H)$ as a scroll. The assertion in the other direction is obvious.

We turn now to the assertion 2) of the lemma. The hypotheses imply that $\mathcal{O}_X(K_X + H)$ is nef but not big, i.e., that $(K_X + H)^2 = 0$. Since $\mathcal{O}_X(K_X + H) \neq \mathcal{O}_X$ (otherwise $\mathcal{O}_X(K_X) = \mathcal{O}_X(-H)$ and hence $q = h^1(\mathcal{O}_X) = h^1(\mathcal{O}_X(K_X)) = h^1(\mathcal{O}_X(-H)) = 0$), the linear system $|K_X + H|$ induces a morphism whose image is a curve. More precisely, there is a morphism

$$\pi : X \longrightarrow B \quad (9.6)$$

onto a smooth curve B with connected fibres and a base point free line bundle $\mathcal{O}_B(D)$ on B such that $\mathcal{O}(K_X + H) = \pi^*(\mathcal{O}_B(D))$. From this we obtain the relation

$$K_X + H = \deg(D)F$$

in $\text{NS}(X)$, where F stands for the class of a fibre of π . Intersecting with F the above identity, we deduce that $H \cdot F = 2$, i.e., that π in (9.6) is a conic fibration. Thus a general fibre of π is a smooth conic and there is at most a finite number of singular fibres. *A priori*, a singular fibre is either the union of two lines intersecting transversely or a double line. The latter, however, is impossible since $F = 2L$ with L a line leads to $K_X \cdot L = L^2 = -1$ which contradicts $0 = F^2 = 4L^2$. \square

We apply the above result to a surface X subject to the hypotheses of Theorem 9.1

PROPOSITION 9.3. *Let X and V_4 be as in Theorem 9.1 and assume X to be irregular. Then $\mathcal{O}_X(K_X + H)$ is base point free. Furthermore, $\mathcal{O}_X(K_X + H)$ is big and hence $H^1(\mathcal{O}_X(2K_X + H)) = 0$ with the possible exception of X being an ADSR elliptic conic bundle. If such a situation occurs, then X must pass through precisely 32 singular points of V_4 .*

Proof. By Serre duality $H^1(\mathcal{O}_X(2K_X + H)) \cong H^1(\mathcal{O}_X(-(K_X + H)))^*$. According to Lemma 9.2, the latter group is nonzero if X is embedded, either as a scroll, or as a conic bundle. The first possibility implies that X is an elliptic scroll of degree 5, see Lemma 7.8. But such a scroll, as we have seen in the previous section, is contained in hypersurfaces of degree 3, hence it can not occur here. We turn to the second possibility: X is birational to a ruled surface embedded into \mathbb{P}^4 by $\mathcal{O}_X(H)$ as a conic bundle. More precisely, from the proof of Lemma 9.2, 2), the line bundle $\mathcal{O}(K_X + H)$ induces a morphism $\pi : X \rightarrow B$ onto a smooth curve B of genus $q = q(X)$, the irregularity of X , such that the H -degree of the fibres of π is 2. A general fibre of π is a smooth plane conic, while the singular fibres are reduced singular conics.

If X is not minimal, then π factors through a minimal model of X , call it X' , and gives the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ & \searrow \pi & \swarrow \pi' \\ & B & \end{array}$$

where σ is the blow-down of a collection of (-1) -curves on X and X' is a ruled surface over B with π' its structure morphism. The collection of exceptional curves is a choice of one out of the two irreducible components of each reducible fibre of π —these are the only (-1) -curves of X . This implies, in particular, that X is the blow-up of X' at distinct points. Let δ be the number of blown-up points and let $\{C_j\}$ be the collection of the exceptional (-1) -curves on X . Then we write

$$K_X = \sigma^*(K_{X'}) + \Delta,$$

where $K_{X'}$ is the canonical divisor of X' and $\Delta = \sum_{j=1}^{\delta} C_j$ is the sum of the exceptional (-1) -curves blown-down by σ . This implies

$$K_X^2 = K_{X'}^2 - \delta = -8(q-1) - \delta \quad \text{and} \quad c_2 = -4(q-1) + \delta. \quad (9.7)$$

Since $\mathcal{O}_X(H + K_X)$ is composed of a pencil, we also have $(H + K_X)^2 = 0$, hence

$$K_X^2 = -d - 2H \cdot K_X.$$

From this and the first identity in (9.7), it follows that

$$H \cdot K_X = 4(q-1) - \frac{d-\delta}{2}. \quad (9.8)$$

Substituting this and the Chern numbers computed in (9.7) into the double point formula, we deduce the identity

$$d^2 - \frac{15}{2}d - \frac{1}{2}\delta = 16(q-1). \quad (9.9)$$

Since X is a conic bundle, we can associate to $\pi : X \rightarrow B$ the embedding

$$\varphi : B \longrightarrow \mathrm{Gr}(2, \mathbb{P}^4) \quad (9.10)$$

of the base curve B into the Grassmannian $\mathrm{Gr}(2, \mathbb{P}^4)$ of planes in \mathbb{P}^4 , where φ sends a point $b \in B$ to the plane P_b spanned by the conic $F_b = \pi^{-1}(b)$, the fibre of π . Let \mathcal{U} be the pullback under φ of the universal subbundle of $\mathrm{Gr}(2, \mathbb{P}^4)$. It is a rank 3 bundle on B and φ induces the morphism

$$\tilde{\varphi} : \mathbb{P}(\mathcal{U}) \longrightarrow \mathbb{P}^4 \quad (9.11)$$

defined on its projectivization $\mathbb{P}(\mathcal{U})$. The image of $\tilde{\varphi}$ is a 3-fold which, set-theoretically, is the union of the family of planes $\{P_b\}_{b \in B}$. In particular, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{U})}(1) := \tilde{\varphi}^*(\mathcal{O}_{\mathbb{P}^4}(1))$ is determined by the identity

$$\mathcal{U}^* = \rho_*(\mathcal{O}_{\mathbb{P}(\mathcal{U})}(1)) = \pi_*(\mathcal{O}_X(H)), \quad (9.12)$$

where $\rho : \mathbb{P}(\mathcal{U}) \rightarrow B$ is the structure projection. We will need to know the degree of \mathcal{U}^* .

Claim. $\deg(\mathcal{U}^*) = \frac{3d - \delta}{4}.$

To justify the claim, we start by computing the holomorphic Euler characteristic of \mathcal{U}^* in (9.12):

$$\chi(\mathcal{U}^*) = \chi(\mathcal{O}_X(H)) = \frac{H^2 - H \cdot K_X}{2} + \chi(\mathcal{O}_X) = \frac{d - H \cdot K_X}{2} - (q - 1).$$

On the other hand, the Riemann-Roch for \mathcal{U}^* on B gives $\chi(\mathcal{U}^*) = \deg(\mathcal{U}^*) - 3(q - 1)$. Putting the two expressions for $\chi(\mathcal{U}^*)$ together, we deduce

$$\deg(\mathcal{U}^*) = \frac{d - H \cdot K_X}{2} + 2(q - 1).$$

This and the expression for $H \cdot K_X$ in (9.8) imply the equality of the claim.

Set

$$d' := \frac{3d - \delta}{4}. \quad (9.13)$$

The geometric meaning of d' is two-fold:

- 1) If $\varphi' : B \hookrightarrow \mathbb{P}^9$ is the composition of φ with the Plücker embedding of $\mathrm{Gr}(2, \mathbb{P}^4)$ then d' is the degree of the image $B' = \varphi'(B)$.
- 2) If V is the image of $\tilde{\varphi}$, then $d' = \deg(V)$.

It will be more convenient at this point to express the double point formula (9.9) in terms of d and d' :

$$d^2 - 9d + 2d' = 16(q - 1). \quad (9.14)$$

We also bring in the ndp formula (9.3) to obtain an upper bound for d'

$$\begin{aligned} d^2 &= 4(H^2 + H \cdot K_X) + \deg(Z) \\ &= 4\left(d + 4(q - 1) - \frac{d - \delta}{2}\right) + \deg(Z) \\ &= 16(q - 1) + 8(d - d') + \deg(Z), \end{aligned}$$

where the second equality uses (9.8) and the last one (9.13). This expression for d^2 and the identity (9.14) give

$$\deg(Z) = d + 6d'. \quad (9.15)$$

From this, the upper bound $\deg(Z) \leq 45$ in (9.5), and $d \geq 5$, we deduce that $d' \leq 6$. This upper bound tells us that the curve B' , the image of φ , spans a subspace \mathbb{P}^N of dimension $3 \leq N \leq 5$. The value $N = 2$ is excluded; indeed, if B' spans a plane, then that plane either intersects $\text{Gr}(2, \mathbb{P}^4)$ along B' or it is contained in the Grassmannian $\text{Gr}(2, \mathbb{P}^4)$. The first possibility means that B' is a conic and hence, $q = 0$, while the second possibility tells us that all planes P_b , $b \in B$, intersect along a line, call it l ; but then $l \cong \mathbb{P}^1 \subset X$ is a multi-section of $\pi : X \rightarrow B$ and this forces q to be zero again. Furthermore, if $N = 5$, then $d' = 6$ and B must be an elliptic curve, *i.e.*, $q = 1$. Substituting these values in (9.14), we obtain

$$d^2 - 9d + 12 = 0$$

which has no integer solutions. Thus $N = 3$ or 4.

If $N = 4$, then $d' = 5$ or 6. The first possibility implies again that $q = 1$ and the formula (9.14) becomes $d^2 - 9d + 10 = 0$ with no integer solution. The second possibility, $d' = 6$, implies that $q = 1, 2$. The first value has been ruled out in the discussion of the case $N = 5$. As for the second, the formula (9.14) becomes $d^2 - 9d - 4 = 0$ with no integer solutions.

Thus $N = 3$ is the only admissible value, while $d' = 4, 5$ or 6. For $d' = 6$, the Castelnuovo upper bound on genus gives $q \leq 4$. Only $q = 4$ is compatible with (9.14), leading to the equation $d^2 - 9d - 36 = 0$. Hence $d = 12$. Substituting into (9.15), we obtain the contradiction

$$45 \geq \deg(Z) = 12 + 36 = 48.$$

For $d' = 5$, the Castelnuovo upper bound implies $q = 1, 2$. The first value was already discarded in the case $N = 4$, while the second value substituted into (9.14) gives $d^2 - 9d - 6 = 0$ with no integral solutions.

Thus we are left with $d' = 4$ and hence $q = 1$. These values substituted in (9.14) yield $d^2 - 9d + 8 = 0$ with the integer solution $d = 8$. We now go to (9.13) to deduce

$$K_X^2 = -\delta = -8.$$

This together with (9.8) imply $H \cdot K_X = 0$. Thus X is an ADSR elliptic conic bundle. Furthermore, from the formula (9.15) it follows that $\deg(Z) = 32$. This completes the proof of the proposition. \square

As we have mentioned in the discussion following the statement of Theorem 9.1, we do not know if an ADSR elliptic conic bundle is contained in a quartic hypersurface with isolated ordinary double points only. On the other hand, such a surface, by its very definition, is contained in a distinguished quartic whose singular locus is 2-dimensional. This implicitly appears in the works [1] and [28]. In the proof of Proposition 9.3 this distinguished quartic is V , the image of the morphism $\tilde{\varphi} : \mathbb{P}(\mathcal{U}) \rightarrow \mathbb{P}^4$ in (9.11). The following statement summarizes the properties of the vector bundle \mathcal{U} and its relation to the geometry of V .

PROPOSITION 9.4. *Let \mathcal{U}^* be the vector bundle defined in (9.12) and let V be the image of the morphism $\tilde{\varphi} : \mathbb{P}(\mathcal{U}) \rightarrow \mathbb{P}^4$ in (9.11). Then*

1) \mathcal{U}^* has the form

$$\mathcal{U}^* = \mathcal{O}_B \oplus \mathcal{F}^*, \quad (9.16)$$

where \mathcal{F}^* is a rank 2 bundle on B fitting into the exact sequence

$$0 \longrightarrow \mathcal{O}_B(D) \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{O}_B(D') \longrightarrow 0, \quad (9.17)$$

with $\mathcal{O}_B(D)$ and $\mathcal{O}_B(D')$ being line bundles of degree 2.

- 2) $V \subset \mathbb{P}^4$ is a hypersurface of degree 4. It is a cone with vertex $[v]$, the image of the section of $\mathbb{P}(\mathcal{U})$ corresponding to the trivial summand in the direct sum decomposition (9.16).
- 3) The summand \mathcal{F}^* in (9.16) determines a distinguished divisor $\mathbb{P}(\mathcal{F})$ of $\mathbb{P}(\mathcal{U})$. Its image S under $\tilde{\varphi}$ is a base of the cone V . In particular, S is a quartic surface with the singular locus $\text{Sing}(S)$ consisting of either one or two (skew) lines in \mathbb{P}^3 . The latter possibility occurs when the exact sequence (9.17) splits.
- 4) The singular locus $\text{Sing}(V)$ of V is the cone over $\text{Sing}(S)$ with vertex at $[v]$. In particular, set-theoretically, it is composed of either one or two planes depending on whether or not the sequence (9.17) is non split.

Proof. Consider the pullback under the morphism φ in (9.10) of the dual of the universal sequence on $\text{Gr}(2, \mathbb{P}^4)$

$$0 \longrightarrow \mathcal{F} \longrightarrow H^0(\mathcal{U}^*) \otimes \mathcal{O}_B \longrightarrow \mathcal{U}^* \longrightarrow 0. \quad (9.18)$$

This implies that $H^0(\mathcal{F}) = 0$. The subbundle $\mathcal{F} \subset H^0(\mathcal{U}^*) \otimes \mathcal{O}_B$ defines the morphism

$$\varphi^\vee : B \longrightarrow \text{Gr}(1, (\mathbb{P}^4)^\vee),$$

dual of the morphism φ in (9.10), where $(\mathbb{P}^4)^\vee = \mathbb{P}(H^0(\mathcal{U}^*))$. Composing φ^\vee with the Plücker embedding $\text{Gr}(1, (\mathbb{P}^4)^\vee) \subset \mathbb{P}(\wedge^2 H^0(\mathcal{U}^*)) \cong (\mathbb{P}^9)^\vee$, we obtain the embedding

$$\psi : B \hookrightarrow (\mathbb{P}^9)^\vee.$$

The image of ψ , as the image of the embedding φ' in the proof of Proposition 9.3, spans a \mathbb{P}^3 . Hence the image of the linear map

$$w : \wedge^2 H^0(\mathcal{U}^*)^* \longrightarrow H^0(\det(\mathcal{F}^*))$$

defining the morphism ψ is 4-dimensional, while $\ker(w)$ is a 6-dimensional subspace of $\wedge^2 H^0(\mathcal{U}^*)^*$. This means that $\mathbb{P}(\ker(w))$ intersects the Grassmann variety of decomposable tensors in $\mathbb{P}(\wedge^2 H^0(\mathcal{U}^*)^*)$ along a subscheme of dimension at least 1 implying that the linear map

$$H^0(\mathcal{U}^*)^* \longrightarrow H^0(\mathcal{F}^*) \quad (9.19)$$

has a non-trivial kernel. Indeed, let $l \wedge l'$ be a nonzero decomposable tensor in $\wedge^2 H^0(\mathcal{U}^*)^*$ lying in the kernel of w . We may assume that the pencil $\text{Span}(l, l')$ injects into $H^0(\mathcal{F}^*)$ under the map in (9.19), since otherwise we are done. Thus we can think of l and l' as two linearly independent global sections of \mathcal{F}^* which are proportional, *i.e.*, $l \wedge l'$ is zero as a section of $\det(\mathcal{F}^*)$. Hence the Koszul sequence of one of these sections gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_B(D) \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{O}_B(D') \longrightarrow 0, \quad (9.20)$$

where $h^0(\mathcal{O}_B(D)) \geq 2$. Hence $\deg(D) \geq 2$. Furthermore, since $\deg(\mathcal{F}^*) = \deg(\mathcal{U}^*) = 4$ and the quotient $\mathcal{O}_B(D')$ must be generated by its global sections and nontrivial (the latter

comes from $H^0(\mathcal{F}) = 0$), we deduce $\deg(D) = \deg(D') = 2$. This and the exact sequence (9.20) imply $h^0(\mathcal{F}^*) = 4$. Since $H^0(\mathcal{U}^*)^* \cong H^0(\mathcal{O}_X(H))$ is 5-dimensional, we deduce that the kernel in (9.19) is nontrivial.

Considering the dual of (9.18), we see that the kernel in (9.19) is $H^0(\mathcal{U})$. Now, from $H^0(\mathcal{U}) \neq 0$ and the global generation of \mathcal{U}^* , we deduce the direct sum decomposition

$$\mathcal{U}^* \cong \mathcal{O}_B \oplus \mathcal{G}. \quad (9.21)$$

This together with (9.18) implies that the direct summand \mathcal{G} fits into the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow H^0(\mathcal{G}) \otimes \mathcal{O}_B \longrightarrow \mathcal{G} \longrightarrow 0. \quad (9.22)$$

It remains to identify \mathcal{G} with \mathcal{F}^* . To this end, we go back to the morphism $\varphi' : B \rightarrow \mathbb{P}^9$ in the proof of Proposition 9.3 and recall that its image spans a \mathbb{P}^3 . This means that the linear map

$$\wedge^3 H^0(\mathcal{U}^*) \longrightarrow H^0(\det(\mathcal{U}^*))$$

has the image of dimension 4 or, equivalently, the kernel of dimension 6. This together with the decomposition in (9.21) implies that the linear map

$$\wedge^2 H^0(\mathcal{G}) \longrightarrow H^0(\det(\mathcal{G}))$$

has the kernel, call it W , of dimension at least 2. Combining this and the second exterior power of (9.22) gives $h^0(\mathcal{F} \otimes \mathcal{G}) = h^0(\text{End}(\mathcal{F}^*, \mathcal{G})) \geq 2$. It follows that a general morphism $\mathcal{F}^* \rightarrow \mathcal{G}$ is an isomorphism. Thus $\mathcal{G} \cong \mathcal{F}^*$. Substituting into (9.21), we deduce the decomposition asserted in (9.16). Furthermore, we have $h^0(\text{End}(\mathcal{F}^*, \mathcal{F}^*)) \geq 2$. A nontrivial endomorphism of \mathcal{F}^* gives rise to the exact sequence (9.17).

The remaining assertions of the proposition are obvious geometric analogues of the properties of \mathcal{U}^* (resp. \mathcal{F}^*) in 1). \square

Remark 9.5. Let X be a smooth surface in \mathbb{P}^4 such that

- X is birational to an irregular ruled surface,
- $m_X = 4$ and X is contained in a quartic hypersurface with only nondegenerate isolated singularities,
- the degree of X in \mathbb{P}^4 is not 8.

Then from the proof of Proposition 9.3 it follows that the fibres of X are embedded as curves of degree at least 3.

9.2. The study of $H^1(\mathcal{J}_Z(K_X + 4H))$

Our result here is as follows.

PROPOSITION 9.6. *Let X and V_4 be as in Theorem 9.1 and assume X to be irregular. Then $H^1(\mathcal{J}_Z(K_X + 4H)) = 0$ with a possible exception of X being an ADSR elliptic conic bundle. If such a situation occurs, then $h^1(\mathcal{J}_Z(K_X + 4H)) = 1$ and $Z = X \cap \text{Sing}(V_4)$ is a subset of 32 nodes of V_4 .*

We assume the nonvanishing of the cohomology group $H^1(\mathcal{J}_Z(K_X + 4H))$. The identification

$$H^1(\mathcal{J}_Z(K_X + 4H))^* = \text{Ext}^1(\mathcal{J}_Z(4H), \mathcal{O}_X)$$

provided by the Serre duality, gives rise to a nontrivial extension sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_Z(4H) \longrightarrow 0. \quad (9.23)$$

We may assume the sheaf \mathcal{E} in the middle of this sequence to be locally free. The Bogomolov semistability condition for this sheaf reads

$$0 \leq 4c_2(\mathcal{E}) - c_1^2(\mathcal{E}) = 4 \deg(Z) - 16H^2 = 4(\deg(Z) - 4d). \quad (9.24)$$

From this and the upper bound $\deg(Z) \leq 45$, it follows that \mathcal{E} is Bogomolov unstable provided $d \geq 12$. For this reason, the proof of Proposition 9.6 is naturally divided into two parts:

- the first part rules out the case $d \geq 12$ by examining the geometric consequences of the Bogomolov instability of \mathcal{E} ;
- the second part deals with the remaining values $5 \leq d \leq 11$ for the degree of X .

First part: $d \geq 12$. We begin by recording some geometric consequences of the Bogomolov instability condition $\deg(Z) < 4d$ for the sheaf \mathcal{E} in (9.23).

Let $\mathcal{O}_X(A)$ be the maximal Bogomolov destabilizing subsheaf of \mathcal{E} . Combining the inclusion $\mathcal{O}_X(A) \rightarrow \mathcal{E}$ with the defining extension sequence (9.23) gives the diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{O}_X(A) & & & & \\
 & & \downarrow & \searrow & & & \\
 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{J}_Z(4H) \longrightarrow 0 \\
 & & \searrow & & \downarrow & & \\
 & & & & \mathcal{J}_{Z'}(E) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array} \quad (9.25)$$

where Z' is a 0-dimensional subscheme of X and $\mathcal{J}_{Z'}$ is its ideal sheaf. The Bogomolov destabilizing condition tells us that the divisor

$$B := A - 2H$$

is in $N^+(X)$, the positive cone of X . In particular,

$$A = 2H + B \quad \text{and} \quad E = 2H - B. \quad (9.26)$$

The first formula implies that the slanted arrows in the above diagram are nonzero. From this it follows that E is an effective, nonzero divisor and that those arrows are defined by multiplication by a section $e \in H^0(\mathcal{O}_X(E))$ corresponding to E . In particular, from the upper (resp. lower) slanted arrow, we deduce that Z (resp. Z') is contained in E . On the

other hand we also know that the linear system $|\mathcal{J}_Z(3H)|$ is base point free outside Z . Hence there is a reduced irreducible divisor $C \in |3H|$ containing Z . Therefore, $Z \subset C \cdot E$ and thus subject to the estimate

$$\deg(Z) \leq 3H \cdot E. \quad (9.27)$$

Computing the second Chern number $c_2(\mathcal{E}) = \deg(Z)$ from the vertical sequence in (9.25) gives

$$\deg(Z) = A \cdot E + \deg(Z') = (4H - E) \cdot E + \deg(Z') = 4H \cdot E - E^2 + \deg(Z'). \quad (9.28)$$

Together with the inequality (9.27), this expression implies

$$E^2 \geq H \cdot E + \deg(Z').$$

This inequality acquires more geometry by observing that the divisor $E - H$ is effective¹⁷, allowing us to write $E = H + R$, where R is an effective divisor. Combining this and the second formula in (9.26), we have

$$H = B + R. \quad (9.29)$$

In addition, by substituting the formulas from (9.26) into (9.28), we obtain the identity

$$\deg(Z) = A \cdot E + \deg(Z') = (2H + B) \cdot (2H - B) + \deg(Z') = 4d - B^2 + \deg(Z'), \quad (9.30)$$

which, together with the bound $\deg(Z) \leq 45$, enables us to control the degree d and the intersection number $H \cdot R$.

LEMMA 9.7. *If $d \geq 12$, then $H \cdot R \leq 2$, i.e., R is empty, a line, or a conic.*

Proof. Assume that $H \cdot R \geq 3$. By the Hodge index we have

$$B^2 \leq \frac{(H \cdot B)^2}{d} = \frac{(H \cdot (H - R))^2}{d} = \frac{(d - H \cdot R)^2}{d} \leq \frac{(d - 3)^2}{d}$$

where the last inequality uses the fact that $\frac{(d - H \cdot R)^2}{d}$ is a decreasing function of $H \cdot R$ on the interval $[3, d]$. Substituting this upper bound for B^2 in (9.30) gives the estimate

$$45 \geq \deg(Z) = 4d - B^2 + \deg(Z') \geq 4d - \frac{(d - 3)^2}{d} + \deg(Z') = 3d + 6 - \frac{9}{d} + \deg(Z'). \quad (9.31)$$

From this it follows that the only admissible values for d are 12 and 13. Furthermore, the ndp formula (9.3) yields the divisibility condition

$$\deg(Z) \equiv d^2 \pmod{8}. \quad (9.32)$$

For $d = 13$, it implies $\deg(Z) \leq 41$, hence (9.31) becomes

$$41 \geq \deg(Z) \geq 3 \cdot 13 + 6 - \frac{9}{13} + \deg(Z') > 44$$

which is absurd. The same argument rules out the other admissible value as well. \square

¹⁷This is seen by tensoring (9.25) with $\mathcal{O}_X(-H)$ and showing that $H^0(\mathcal{J}_{Z'}(E - H)) \neq 0$; the argument is exactly the same as in the proof of Lemma 7.3.

Next we examine the three possibilities for R provided by the previous lemma.

If $R = 0$, then $B = H$ and $Z' = 0$ and the formula (9.30) becomes $45 \geq \deg(Z) = 3d$. Hence $12 \leq d \leq 15$. But none of these values satisfies the divisibility condition (9.32).

If R is a line, then $B = H - R$ and $B^2 = d - 2 + R^2$. Substituting the self-intersection number into (9.30) gives

$$45 \geq \deg(Z) = 3d + 2 - R^2 + \deg(Z').$$

Hence $d = 12, 13$ or 14 . The last two values together with the divisibility condition in (9.32) imply that $\deg(Z) = 41$ and 44 respectively. But both values force $R^2 = \deg(Z') = 0$. Thus an irregular X is a scroll and this is ruled by Remark 9.5. Therefore, we are left with $d = 12$ and $\deg(Z) = 40$. However, the inequality in (9.27) now reads

$$40 \leq 3H \cdot E = 3H \cdot (H + R) = 36 + 3 = 39$$

which is absurd.

If R is a conic, then $H \cdot B = 10$ and $B^2 = d - 4 + R^2$. Substituting into (9.30) gives

$$45 \geq \deg(Z) = 3d + 4 - R^2 + \deg(Z').$$

This together with the divisibility condition (9.32) implies $\deg(Z) = 40$, $d = 12$, and $\deg(Z') = R^2 = 0$. The last equality together with $H \cdot R = 2$ tells us that an irregular surface X is a conic bundle and this is impossible by Remark 9.5. This completes the treatment of the case $d \geq 12$ and thus proves the vanishing of $H^1(\mathcal{J}_Z(K_X + 4H))$ for these values of d .

Second part: $5 \leq d \leq 11$. Though the argument comes down to a case by case consideration, there is a basic feature that is common to all of them. This aspect will be explained next. We begin by recalling that the values of the degree d control the values of $\deg(Z)$ via the divisibility condition (9.32), $\deg(Z) \equiv d^2 \pmod{8}$. Hence we obtain

$$\deg(Z) = \begin{cases} 8k + 1, & \text{if } d \in \{5, 7, 9, 11\} \\ 8k + 4, & \text{if } d \in \{6, 10\} \\ 8k, & \text{if } d = 8 \end{cases} \quad (9.33)$$

where $1 \leq k \leq 5$, in view of the upper bound $\deg(Z) \leq 45$. Thus for every value of d we obtain a list of admissible values for $\deg(Z)$. These values are divided into two types according to whether the sheaf \mathcal{E} in the extension sequence (9.23) is semistable or unstable in the sense of Bogomolov.

When \mathcal{E} is unstable, all the possibilities are discarded by exploiting the decomposition $H = B + R$ in (9.29) and by observing that the divisor R has to move in a linear system in order for the hypersurface V_4 to have isolated singularities.

In the semistable case there are two basic ingredients:

- we check that X is birational to a ruled surface of irregularity q ,
- the two conditions, X birational to a ruled surface and $\mathcal{O}_X(K_X + H)$ base point free and big, exclude all but one possibility: X is an ADSR elliptic conic bundle, see Proposition 9.3.

With these general guidelines in mind, we proceed with the second part of the proof according to the possible values of the degree, $5 \leq d \leq 11$.

• The case $d = 11$. According to (9.33), we have $\deg(Z) = 8k + 1 \leq 41$. This upper bound insures that the sheaf \mathcal{E} in the middle of the extension sequence (9.23) is still Bogomolov unstable, see (9.24). Thus the argument used in the case $d \geq 12$ applies and we obtain the decomposition

$$H = B + R$$

as in (9.29). This implies

$$H \cdot B = H \cdot (H - R) = d - H \cdot R \leq d$$

and, hence, by the Hodge index, $B^2 \leq d$. Substituting into (9.30) gives

$$\deg(Z) = 4d - B^2 + \deg(Z') \geq 3d + \deg(Z') \geq 3d = 33. \quad (9.34)$$

Thus $\deg(Z) = 33$ or 41 . Furthermore, for the first value all the above inequalities must be equalities and hence R and Z must both be zero. It follows that $E = H$ and that the vertical sequence in (9.25) takes the form

$$0 \longrightarrow \mathcal{O}_X(3H) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(H) \longrightarrow 0. \quad (9.35)$$

We have already encountered a similar situation in the proof of Lemma 7.5 and we use the same argument here. Namely, recall the identification

$$H^0(\mathcal{E}(-H)) \cong H^0(\mathcal{J}_Z(3H)) \quad (9.36)$$

resulting from the defining extension sequence (9.23) tensored with $\mathcal{O}_X(-H)$. On the other hand, the destabilizing sequence (9.35) gives

$$0 \longrightarrow H^0(\mathcal{O}_X(2H)) \longrightarrow H^0(\mathcal{E}(-H)) \longrightarrow H^0(\mathcal{O}_X).$$

This together with the fact that the above inclusion $H^0(\mathcal{O}_X(2H)) \hookrightarrow H^0(\mathcal{E}(-H))$ must be proper, implies that the arrow on the right must be onto. Hence $H^0(\mathcal{O}_X(2H)) \hookrightarrow H^0(\mathcal{E}(-H))$ is a codimension 1 subspace of $H^0(\mathcal{E}(-H))$. This and the isomorphism (9.36) tell us that the subspace $e H^0(\mathcal{O}_X(2H))$, where $e \in H^0(\mathcal{O}_X(H))$ is a section defining the divisor E , is a codimension 1 subspace of $H^0(\mathcal{J}_Z(3H))$.

Next recall that the space $H^0(\mathcal{J}_Z(3H))$ contains the 5-dimensional subspace W , spanned by the restrictions to X of the partial derivatives of a homogeneous polynomial, call it f , defining the quartic hypersurface V_4 that contains X . The above discussion implies that the subspace $e H^0(\mathcal{O}_X(2H))$ intersect W along a subspace of codimension at most 1. This means that we can choose homogeneous coordinate functions X_i , $i = 0, \dots, 4$, on \mathbb{P}^4 so that

$$\frac{\partial f}{\partial X_i} = h \gamma_i, \quad \text{for } i = 0, \dots, 3,$$

where $\gamma_i \in H^0(\mathcal{O}_{\mathbb{P}^4}(2))$ and $h \in H^0(\mathcal{O}_{\mathbb{P}^4}(1))$ is the linear form corresponding to the section e under the identification $H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \cong H^0(\mathcal{O}_X(H))$. From this it follows that V_4 is singular along the subvariety $(h = \frac{\partial f}{\partial X_4} = 0)$ which contradicts the assumption that V_4 has only isolated singularities.

Notice that the above argument remains valid as long as the space $H^0(\mathcal{J}_{Z'}(R))$ —where the cokernel of $H^0(\mathcal{O}_X(A - H) \rightarrow H^0(\mathcal{E}(-H)))$ lives—is 1-dimensional. This will be our tool

to rule out all the remaining cases whenever the sheaf \mathcal{E} in (9.23) is Bogomolov unstable. We give a full account of this for $\deg(Z) = 41$.

From the identity

$$B^2 = 4d - \deg(Z) + \deg(Z') = 44 - 41 + \deg(Z') = 3 + \deg(Z')$$

we deduce the inequality $B^2 \geq 3$. This and the Hodge index give $H \cdot B \geq 6$ or, equivalently,

$$H \cdot R = H \cdot (H - B) = d - H \cdot B \leq 5.$$

From the inequality (9.28) we also obtain the lower bound

$$H \cdot R \geq \frac{\deg(Z)}{3} - d = \frac{41}{3} - 11,$$

i.e., $H \cdot R \geq 3$, and proceed according to the possible values of

$$H \cdot R = 3, 4, \text{ or } 5.$$

As we have said above, the main idea is, as in the case of $\deg(Z) = 33$, to show that the space $H^0(\mathcal{J}_{Z'}(R))$ is 1-dimensional. For this we will also need the formula

$$\deg(Z') = B^2 - 3 = (H - R)^2 - 3 = d - 3 - 2H \cdot R + R^2 = 8 - 2H \cdot R + R^2 \quad (9.37)$$

which relates $\deg(Z')$ to the intersection numbers $H \cdot R$ and R^2 .

1) $H \cdot R = 3$. We wish to analyse the possibility $h^0(\mathcal{O}_X(R)) \geq 2$. Let R_0 (resp. R') be the fixed (resp. moving) part of the linear system $|R|$. If $R_0 \neq 0$, then $H \cdot R' \leq 2$ and hence, by Hodge index, $R'^2 \leq 0$. Since the linear system $|R'|$ has at most a 0-dimensional base locus, we deduce the equality $R'^2 = 0$. But this means that X is either a scroll or a conic bundle and, in view of Remark 9.5, neither possibility is allowed. Thus we may assume that the linear system $|R|$ has at most a 0-dimensional base locus. This and the Hodge index imply $R^2 = 0$. Hence $|R|$ is base point free. At the same time, observe that $h^0(\mathcal{O}_X(B)) = h^0(\mathcal{O}_X(H - R)) \geq 1$, *i.e.*, B is effective and $h^0(\mathcal{O}_X(H - B)) = h^0(\mathcal{O}_X(R)) \geq 2$. Since B can not be a line, we deduce that if R moves on X , then $h^0(\mathcal{O}_X(R)) = 2$. From the formula (9.37), we also find that $\deg(Z') = 2$. Hence

$$h^0(\mathcal{J}_{Z'}(R)) \leq h^0(\mathcal{O}_X(R)) - 1 = 1$$

and from here on we conclude as in the case $\deg(Z) = 33$.

2) $H \cdot R = 4$. The argument follows the same pattern as in the previous case. Write $R = R_0 + R'$ as above. If $R_0 \neq 0$, then $H \cdot R' \leq 3$ and we deduce that $R'^2 = 0$. Hence, $|R'|$ is a base point free pencil inducing the morphism

$$f : X \longrightarrow \mathbb{P}^1. \quad (9.38)$$

The fibres of f must be connected (otherwise we are back to the situation of X being a scroll or a conic bundle) with $H \cdot R' = 3$ (resp. $H \cdot R_0 = 1$) and hence, f is an elliptic fibration¹⁸ with fibres embedded by $\mathcal{O}_X(H)$ as plane curves of degree 3. The union of the planes spanned

¹⁸The other possibility is that f has rational fibres, but then X is rational.

by the fibres of f form a hypersurface, call it V . The degree of this hypersurface is subject to the inequality

$$\deg(V) \leq \deg(f_*(\mathcal{O}_X(H)))$$

Setting $f_*(\mathcal{O}_X(H)) = \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(a_i)$, we obtain

$$\deg(V) \leq \sum_{i=1}^3 a_i = h^0(f_*(\mathcal{O}_X(H))) - 3 = h^0(\mathcal{O}_X(H)) - 3 = 5 - 3 = 2,$$

contradicting the assumption on the smallest degree of a hypersurface containing X .

We have just shown that the linear system $|R|$ has at most a 0-dimensional base locus. By the Hodge index $R^2 \leq 1$. Hence either $R^2 = 0$ or $R^2 = 1$.

If $R^2 = 0$, then $|R|$ is base point free and induces a morphism as in (9.38). We may again assume that the fibres of this morphism are connected, since otherwise X is either a scroll or a conic fibration. In view of the degree $H \cdot R = 4$ of R , the morphism $f : X \rightarrow \mathbb{P}^1$ is either an elliptic fibration or a fibration by plane curves of degree 4. Hence

$$K_X \cdot R = 0 \quad \text{or} \quad K_X \cdot R = 4. \quad (9.39)$$

The ndp formula tells us that $K_X \cdot H = 9$ and together with (9.39) implies

$$K_X \cdot B = K_X \cdot (H - R) = 9 \quad \text{or} \quad 5.$$

Since $B^2 = (H - R)^2 = 11 - 2H \cdot R = 11 - 2 \cdot 4 = 3$, we obtain

$$B^2 + K_X \cdot B = 12 \quad \text{or} \quad 8 \quad (9.40)$$

which is the degree of the dualizing sheaf of B . But B is as a divisor of degree $d_B = H \cdot B = 7$ contained in a plane¹⁹. Hence the degree of its dualizing sheaf verifies

$$d_B(d_B - 3) = 7 \cdot 4 = 28,$$

which does not match any of the values in (9.40).

If $R^2 = 1$, then $|R|$ must have a unique base point and we may assume that a general member of the linear system is either an elliptic curve or a smooth plane curve of degree 4. These possibilities lead to $K_X \cdot R = -1$ and $K_X \cdot R = 3$ respectively. Together with the identity $K_X \cdot H = 9$ (the ndp formula), they imply

$$K_X \cdot B = 10 \quad (\text{resp.} = 6).$$

But $B^2 = (H - R)^2 = 4$, hence the degree of the dualizing sheaf of B verifies

$$B^2 + K_X \cdot B = 14 \quad (\text{resp.} = 10)$$

and, as in the case $R^2 = 0$, neither value agrees with $d_B(d_B - 3) = 28$.

3) $H \cdot R = 5$. Then $H \cdot B = 6$, $B^2 = 3$, and $R^2 = 2$. Writing $R = R_0 + R'$ as before, we see that $R_0 \neq 0$ reduces to the cases previously considered. So we may assume that the linear system $|R|$ has at most a 0 dimensional base locus. Furthermore, from $R^2 = 2$ it follows that a general member of $|R|$ is irreducible. If, in addition, a general curve in the linear system is

¹⁹This follows from $h^0(\mathcal{O}_x(H - B)) = h^0(\mathcal{O}_X(R)) = 2$.

not contained in a hyperplane, then by the Castelnuovo upper bound on the genus of curves in \mathbb{P}^4 , a general member of $|R|$ is a smooth curve of genus 1. This and the adjunction for R give $K_X \cdot R = -R^2 = -2$ implying that X is birational to a ruled surface with $q = 1$. From the ndp formula in (9.3), we have $H \cdot K_X = 9$ and obtain $K_X^2 = -17$.

Next we consider the line bundle $\mathcal{O}_X(K_X + 2R)$. From the Riemann-Roch,

$$h^0(\mathcal{O}_X(K_X + 2R)) = (K_X + 2R) \cdot R = R^2 = 2.$$

On the other hand, $(K_X + 2R)^2 = K_X^2 + 4K_X \cdot R + 4R^2 = K_X^2 = -17$. Hence the linear system $|K_X + 2R|$ has a fixed part $F \neq 0$. In particular, there is a dense open subset F' of F such that through every point $x \in F'$ passes an irreducible curve $R_x \in |R|$. But x is a base point of $\mathcal{O}_X(K_X + 2R)$ and by [29, Theorem 1], there is an irreducible curve $C \subset X$ passing through x with the property $C \cdot R_x = C \cdot R = 0$. Hence the curves C and R_x have a common component. Since both curves are irreducible, we obtain $C = R_x$ and, therefore, $0 = C \cdot R_x = R_x^2 = R^2 = 2$, an obvious contradiction.

Now we know that a general curve in the linear system $|R|$ is contained in a hyperplane, hence that $H^0(\mathcal{O}_X(B)) = H^0(\mathcal{O}_X(H - R)) \neq 0$. This tells us that B is an effective divisor. Furthermore, $h^0(\mathcal{O}_X(H - B)) = h^0(\mathcal{O}_X(R)) \geq 2$ tells us that equality must hold, *i.e.*, B is a plane curve subject to $d_B = H \cdot B = 6$ and $B^2 = 3$. Hence

$$B^2 + K_X \cdot B = d_B(d_B - 3) = 18.$$

It follows that $K_X \cdot B = 18 - B^2 = 15$. But then

$$15 = K_X \cdot B = K_X \cdot (H - R) = K_X \cdot H - K_X \cdot R \leq 9 - (-2 - R^2) = 9 + 4 = 13$$

gives an obvious contradiction. This completes the case $d = 11$.

- The case $d = 10$. As in the case $d = 11$, we obtain two possible values for $\deg(Z)$: 36 and 44. For the first value, the sheaf \mathcal{E} in (9.23) is Bogomolov unstable. The considerations are the same as in the case $d = 11$ and we leave the details to the reader. For the latter, *i.e.*, if $\deg(Z) = 44$, then we follow the plan outlined in the paragraph prior to the case $d = 11$.

We begin by showing that X is birational to a ruled surface. The ndp formula reads

$$H^2 + H \cdot K_X = 14,$$

hence $H \cdot K_X = 4$. Combining this with the Hodge Index, gives $K_X^2 \leq 1$. Substituting this upper bound into the double point formula yields

$$-20 = 2K_X^2 - 12\chi \leq 2 - 12\chi.$$

Hence $\chi \leq 1$ and we obtain

$$K_X^2 = -10 + 6\chi \leq -4. \tag{9.41}$$

We are now ready to show that X is birational to a ruled surface. Since we are assuming that X is irregular, it is enough to show that the Kodaira dimension of X is negative. Assume the opposite and let X_0 be the minimal model of X with $\sigma : X \rightarrow X_0$ the sequence of blow down maps. We write

$$K_X = \sigma^* K_0 + \Delta,$$

where K_0 is the canonical divisor of X_0 and Δ is the exceptional divisor composed of the blown-down curves. Since $K_0^2 \geq 0$, the estimate in (9.41) tells us that σ is the composition of at least four blow-downs. It follows that Δ has at least four irreducible components, *i.e.*, that $H \cdot \Delta \geq 4$. This gives

$$4 = H \cdot K_X = H \cdot \sigma^* K_0 + H \cdot \Delta \geq H \cdot \sigma^* K_0 + 4 \quad (9.42)$$

or, equivalently, $H \cdot \sigma^* K_0 \leq 0$. Since $\sigma^* K_0$ is nef, it follows that $H \cdot \sigma^* K_0 = 0$ and X is of Kodaira dimension zero. By the Enriques-Kodaira classification, we must have $\chi = 0$. Returning to the equality in (9.41), we obtain $K_X^2 = -10$. But then Δ has at least ten irreducible components and the estimate in (9.42) gives $H \cdot \sigma^* K_0 \leq -6$ which is impossible. Hence X is birational to a ruled surface and $\chi(\mathcal{O}_X) = 1 - q$, which, substituted into the equality (9.41), gives

$$K_X^2 = -10 + 6\chi(\mathcal{O}_X) = -10 - 6(q - 1). \quad (9.43)$$

We pursue by studying the adjoint divisor $K_X + H$. We have, using (9.43),

$$(K_X + H)^2 = K_X^2 + 2H \cdot K_X + H^2 = -10 - 6(q - 1) + 2 \cdot 4 + 10 = 8 - 6(q - 1)$$

and recalling that $\mathcal{O}_X(K_X + H)$ is base point free, see Proposition 9.3, we deduce that either $q = 1$ or $q = 2$.

If $q = 2$, then $(K_X + H)^2 = 2$. By Riemann-Roch, we have

$$h^0(\mathcal{O}_X(K_X + H)) = \frac{(K_X + H) \cdot H}{2} + \chi(\mathcal{O}_X) = 7 - 1 = 6.$$

But, according to Proposition 9.3, the linear system $|K_X + H|$ is base point free and hence defines a morphism $X \rightarrow \mathbb{P}^5$ whose image is a surface of degree at most $(K_X + H)^2 = 2$ and this is impossible.

If $q = 1$, then $K_X^2 = -10$ and $(K_X + H)^2 = 8$. The latter and Proposition 9.3 tell us that $|K_X + H|$ is base point free and defines a morphism

$$f : X \longrightarrow \mathbb{P}^6 \quad (9.44)$$

which must be birational onto its image. We also record the projection morphism

$$\pi : X \longrightarrow B \quad (9.45)$$

onto an elliptic curve B with rational fibres. In particular, we wish to understand the degree of the fibres of π with respect to H . For this we look at the double adjoint line bundle $\mathcal{O}_X(2K_X + H)$. By Riemann-Roch

$$h^0(\mathcal{O}_X(2K_X + H)) = \frac{(2K_X + H) \cdot (K_X + H)}{2} = \frac{K_X \cdot (K_X + H) + (K_X + H)^2}{2} = 1.$$

Let D be the divisor defined by a nonzero section of $\mathcal{O}_X(2K_X + H)$. It is a non-zero divisor, since $D^2 = (2K_X + H)^2 = -24$. Furthermore, since $(K_X + H) \cdot D = 2$, and since $|K_X + H|$ is base point free, an irreducible component C of D verifies one of the following possibilities:

$$(i) \quad (K_X + H) \cdot C = 0, \quad (ii) \quad (K_X + H) \cdot C = 1, \quad (iii) \quad (K_X + H) \cdot C = 2. \quad (9.46)$$

The curves C of type (i) are precisely the curves contracted by the morphism f in (9.44). From $(K_X + H) \cdot C = 0$ it follows that C must be a smooth rational curve with $C^2 = H \cdot C - 2$.

This implies that $H \cdot C = 1$ or 2 , since $C^2 \leq 0$. The second value means that X is a conic bundle which is impossible by Remark 9.5. Thus the curves of type (i) are (-1) -curves on X , contracted by $\mathcal{O}_X(K_X + H)$, and mapped onto lines by $\mathcal{O}_X(H)$ in the embedding $X \subset \mathbb{P}^4$. In particular, all these irreducible components of D are contained in the fibres of the projection $\pi : X \rightarrow B$.

The curves C of type (ii) are smooth rational curves (they are lines with respect to the morphism f) with $C^2 = H \cdot C - 3$. Hence $H \cdot C = 1, 2$ or 3 . The last value is impossible, since otherwise 3 is the H -degree of all fibres of π and they all have intersection -1 with D which is impossible. Thus the curves of type (ii) are smooth rational curves on X of self-intersection, either -1 , or -2 . Moreover, they are conics and lines respectively in the embedding by $\mathcal{O}_X(H)$ and are contained in the fibres of the projection $\pi : X \rightarrow B$. Also observe that there are at most 2 distinct such curves in D .

We claim that there is no curve of type (iii). Indeed, all points of such a curve C are fixed points of $\mathcal{O}_X(2K_X + H)$ and according to [29, Theorem 1], through every point $x \in C$ passes an irreducible curve of type (i) or (ii). This is impossible since those curves are rigid.

From the above analysis of the irreducible components of D it follows that

$$D = C_1 + C_2 + D_0,$$

where C_i , $i = 1, 2$, are the curves (not necessarily distinct) of type (ii) and D_0 is the residual part of D composed of curves of type (i). In addition, [29, Theorem 1] stipulates the existence of a unique divisor E_x passing through every point $x \in C_i$, subject to either

$$(K_X + H) \cdot E_x = 0 \quad \text{and} \quad E_x^2 = -1 \tag{a}$$

or

$$(K_X + H) \cdot E_x = 1 \quad \text{and} \quad E_x^2 = 0 \tag{b}$$

We know that the irreducible components of E_x in (a) are the (-1) -curves of type (i) and no such curve passes through a general point of C_i . So for all points x in the complement of some finite subset of C_i , the corresponding divisor E_x passing through x is of type (b). This implies that E_x must be the fibre of π containing C_i . Thus if F is the class of a fibre of the projection π in (9.45), then

$$(K_X + H) \cdot F = 1.$$

or, equivalently, $H \cdot F = 3$. But then

$$D \cdot F = (2K_X + H) \cdot F = 2K_X \cdot F + H \cdot F = -4 + 3 = -1$$

contradicting the fact that D is effective. This completes the proof of the case $d = 10$.

- The case $d = 9$. The possible values for $\deg(Z)$ are 41 and 33, where the first (resp. second) value is ‘stable’ (resp. ‘unstable’). We treat only the stable possibility, *i.e.*,

$$\deg(Z) = 41.$$

From the ndp formula we obtain $H \cdot K_X = 1$. This and the Hodge index imply $K_X^2 \leq 0$. From the double point formula we obtain

$$0 \geq K_X^2 = 6\chi - 7$$

and hence $\chi \leq 1$. Arguing as in the case $d = 10$, $\deg(Z) = 44$, we deduce that X has negative Kodaira dimension or, equivalently, that X is birational to a ruled surface of irregularity q . Thus $\chi = 1 - q$ and $K_X^2 = -7 - 6(q - 1)$.

We proceed with the study of $\mathcal{O}_X(K_X + H)$. The self-intersection

$$(K_X + H)^2 = 11 + K_X^2 = 4 - 6(q - 1)$$

and Proposition 9.3 tell us that the only possibility for the irregularity is $q = 1$. Hence $(K_X + H)^2 = 4$ and $K_X^2 = -7$. Computing the genus of a smooth curve in the adjoint linear system $|K_X + H|$ gives

$$(K_X + H)^2 + (K_X + H) \cdot K_X = 4 + 1 - 7 = -2,$$

i.e., such a curve is rational. But then X can not be irregular.

- The case $d = 8$. The possible values for $\deg(Z)$ are

$$\deg(Z) = 40, 32, \text{ and } 24,$$

where the last one is the only unstable value. It can be treated easily as in the case $d = 11$, $\deg(Z) = 33$, so we turn towards the two remaining stable values.

From the ndp formula we obtain

$$H \cdot K_X = 8 - 2k,$$

where $k = 4$ or 5 . The negativity of the Kodaira dimension follows readily. Hence $\chi = 1 - q$ and the double point formula gives

$$K_X^2 = -8 - 6(q - 1) - 5(4 - k), \quad \text{where } k = 4, 5.$$

We now compute the self-intersection

$$(K_X + H)^2 = -6(q - 1) - (4 - k), \quad \text{where } k = 4, 5.$$

This and Proposition 9.3, according to which $\mathcal{O}_X(K_X + H)$ is base point free, give the only possibility $q = 1$, $k = 4$, $(K_X + H)^2 = 0$. This corresponds to the exceptional possibility of an ADSR elliptic conic bundle.

- The case $d \leq 7$. All those values are incompatible with $\mathcal{O}_X(K_X + H)$ being base point free and big. Indeed, the last condition stipulates $(K_X + H)^2 > 0$, which we rewrite as

$$K_X^2 > -d - 2H \cdot K_X.$$

We substitute this into the double point formula and obtain

$$d^2 - 8d + 12\chi > H \cdot K_X.$$

On the other hand, expressing $H \cdot K_X$ from the ndp formula (9.3) gives

$$H \cdot K_X = \frac{1}{4}(d^2 - 4d - \deg(Z)). \quad (9.47)$$

Putting it together with the previous inequality, we have

$$\frac{3}{4}d^2 - 7d + \frac{1}{4}\deg(Z) + 12\chi > 0. \quad (9.48)$$

The left hand side of this inequality is an increasing function of d for $d \geq 5$. So, for $d \leq 7$, we have $\frac{3}{4}d^2 - 7d \leq -\frac{49}{4}$. Substituting into (9.48) and using $\deg(Z) \leq 45$ give

$$12\chi > \frac{49}{4} - \frac{1}{4} \deg(Z) \geq 1.$$

Hence

$$\chi \geq 1. \quad (9.49)$$

Going back to (9.47) we obtain

$$H \cdot K_X = \frac{1}{4}(d^2 - 4d - \deg(Z)) \leq \frac{1}{4}(7^2 - 4 \cdot 7 - \deg Z) = \frac{1}{4}(21 - \deg Z), \quad (9.50)$$

for $d \leq 7$. In particular, if $d \leq 7$ and $\deg(Z) > 21$, then $H \cdot K_X$ is negative. Hence the Kodaira dimension of X is negative. This together with the estimate (9.49) gives $1 \leq \chi = 1 - q$ meaning that $q = 0$.

If $\deg(Z) \leq 21$, then $\deg(Z) \leq 21 < 4d$, provided $6 \leq d \leq 7$. Hence the values of $\deg(Z) \leq 21$ are unstable and one has the estimate

$$\deg(Z) > 3d, \text{ for } d = 6, 7,$$

coming from the first inequality in (9.34) and the divisibility condition (9.32). This implies the only possibility: $d = 6$ and $\deg(Z) = 20$. Substituting these values into the first equality in (9.50) gives

$$H \cdot K_X = \frac{1}{4}(d^2 - 4d - \deg(Z)) = -2.$$

Hence, as before, $q = 0$.

For the remaining value $d = 5$, the identity (9.47) and the divisibility condition (9.32) give

$$H \cdot K_X = \frac{1}{4}(5 - \deg(Z)) \quad \text{and} \quad \deg(Z) = 8k + 1.$$

Therefore, as before, the Kodaira dimension of X is negative, except possibly in the case $\deg(Z) = 1$. But then the subscheme Z is a single point, call it p , and the group $H^1(\mathcal{J}_p(K_X + 4H)) = 0$, since by [29, Theorem 1], the line bundle $\mathcal{O}_X(K_X + 4H)$ is very ample. This completes the study of cases $5 \leq d \leq 11$.

10. ALBANESE DIMENSION

In this section we consider the Albanese dimension of surfaces lying on small degree hypersurfaces. Since we have already seen that there is no irregular surface lying on a quadric hypersurface and there is only the elliptic scroll on a cubic hypersurface, our examination will focus on hypersurfaces of degree 4 and 5. From now on we set $m = m_X = 4$ or 5 —the smallest degree of a hypersurface containing a smooth surface $X \subset \mathbb{P}^4$ —and we ask the following questions:

- 1) Can X be of Albanese dimension 2?
- 2) If the answer to question 1) is affirmative, can the irregularity of X be bounded?

To investigate these questions we assume X of Albanese dimension 2 and consider the extension sequence (2.7) associated to the cohomology class $\xi \in H^1(\Theta_X(-K_X - (5-m)H))$ arising from a hypersurface of degree m containing X (see Lemma 2.2). Observe: the assumption that the Albanese dimension of X is 2 insures that X has non-negative Kodaira dimension and hence the class $\xi = \delta_X(s)$ in Lemma 2.2 is nonzero. In other words the sequence (2.7) is non split.

10.1. The Albanese dimension of $X \subset \mathbb{P}^4$ with $m_X = 4$

This subsection is devoted to the proof of the following.

THEOREM 10.1. *If $X \subset \mathbb{P}^4$ is a smooth surface with $m_X = 4$, then the Albanese dimension of X is at most 1.*

Proof. Assume that X is subject to the hypothesis of the theorem and that it has the Albanese dimension 2. Since $m = m_X = 4$, the exact sequence (2.7) becomes

$$0 \longrightarrow \mathcal{O}_X(-K_X - H) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0. \quad (10.1)$$

The assumption on the Albanese dimension means that Ω_X is generically generated by global sections. In particular, $q(X) \geq 2$ and K_X is effective. The last property implies that the homomorphism

$$H^0(\mathcal{T}_\xi) \longrightarrow H^0(\Omega_X) \quad (10.2)$$

induced by the epimorphism in (10.1) is an isomorphism: the injectivity follows from the obvious vanishing of $H^0(\mathcal{O}_X(-K_X - H))$ and the surjectivity is insured by the following lemma.

LEMMA 10.2. *If K_X is effective, then the divisor $K_X + H$ is nef and big. In particular, $H^1(\mathcal{O}_X(-K_X - H)) = 0$.*

Proof of Lemma 10.2. Let C be an effective curve such that $(K_X + H) \cdot C \leq 0$. It follows that $K_X \cdot C \leq -H \cdot C < 0$. But K_X is effective, hence $C^2 < 0$ forcing C to be a (-1) -curve, i.e., $C^2 = K_X \cdot C = -1$. Since $H \cdot C \geq 1$, we deduce $(K_X + H) \cdot C \geq 0$ and hence $(K_X + H) \cdot C = 0$. Thus $K_X + H$ is nef. This implies the non-negativity $(K_X + H) \cdot K_X \geq 0$. Hence

$$(K_X + H)^2 = (K_X + H) \cdot K_X + (K_X + H) \cdot H \geq (K_X + H) \cdot H \geq H^2 > 0.$$

□

Once the isomorphism (10.2) has been established, we define the subsheaf $\mathcal{F} \subset \mathcal{T}_\xi$ as the saturation of the subsheaf generated by the global sections of \mathcal{T}_ξ . In particular, the inclusion $\mathcal{F} \subset \mathcal{T}_\xi$ composed with the epimorphism in (10.1) gives the morphism $\varphi : \mathcal{F} \rightarrow \Omega_X$ which is generically surjective. Hence the rank of \mathcal{F} is at least 2. On the other hand (10.1) tells us that the determinant of \mathcal{T}_ξ is $\mathcal{O}_X(-H)$. Hence \mathcal{F} must be of rank 2 and Lemma 2.3 applies to give us the decomposition

$$K_X = L + E \quad (10.3)$$

where $L = c_1(\mathcal{F})$ and $E = c_1(\text{coker}(\varphi))$ is an effective nonzero divisor. Furthermore, L is effective as well, since by definition $H^0(\mathcal{F}) \cong H^0(\Omega_X)$ and \mathcal{F} is generically generated by its

global sections. This and Lemma 2.3, 3), imply that $L = 0$. Hence $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ and therefore, $q(X) = 2$. In addition, the trivial subsheaf \mathcal{F} provides

$$0 \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \longrightarrow \mathcal{T}_\xi \longrightarrow \mathcal{J}_Z(-H) \longrightarrow 0, \quad (10.4)$$

a destabilizing sequence for the sheaf \mathcal{T}_ξ . This sequence will play an important role in a later part of the argument.

In fact, more can be extracted from part 3) of Lemma 2.3. Namely, since $L = 0$ (and $m_X = 4$) that assertion also tells us that we have a unique nonzero global section τ of $\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-H)$ such that its image in $H^0(\mathcal{N}_X(-H))$ is the section $e \cdot s$, where s is the global section of $\mathcal{N}_X(-K_X - H)$ defined by a quartic hypersurface containing X and e is a global section of $\mathcal{O}_X(K_X)$ corresponding to the divisor E in (10.3). Furthermore, putting the normal sequence of X and the Euler sequence of $\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X$ together and tensoring everything with $\mathcal{O}_X(-H)$, we obtain the diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{O}_X(-H) & & & & \\ & & \downarrow & & & & \\ & & H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X & & & & (10.5) \\ & & \downarrow & \searrow \eta & & & \\ 0 & \longrightarrow & \Theta_X(-H) & \longrightarrow & \Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-H) & \longrightarrow & \mathcal{N}_X(-H) \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

from which it follows that the global section τ of $\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-H)$ constructed above comes from a unique element $v \in H^0(\mathcal{O}_X(H))^*$. This vector has the following geometric meaning.

LEMMA 10.3. *Let V_4 be a quartic hypersurface containing X and let $[v]$ be the point of $\mathbb{P}(H^0(\mathcal{O}_X(H))^*)$ corresponding to the vector v above. Then V_4 is a cone with the vertex $[v]$ over a quartic surface S in \mathbb{P}^3 .*

Proof of Lemma 10.3. To establish the asserted relation of V_4 with the vector v we recall that V_4 gives rise to the nonzero global section s of $\mathcal{N}_X(-K_X - H)$. Let Σ be the 1-dimensional part of the zero locus of this section and denote by σ a section of $\mathcal{O}_X(\Sigma)$ corresponding to Σ . Then $s = \sigma s'$, where $s' \in H^0(\mathcal{N}_X(-K_X - H - \Sigma))$ and has a 0-dimensional zero locus $Z_{s'}$. Therefore the Koszul sequence of s'

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s'} \mathcal{N}_X(-K_X - H - \Sigma) \xrightarrow{s' \wedge} \mathcal{J}_{Z_{s'}}(3H - K_X - 2\Sigma) \longrightarrow 0$$

is exact. Tensoring it with $\mathcal{O}_X(K_X + \Sigma)$, we obtain

$$0 \longrightarrow \mathcal{O}_X(K_X + \Sigma) \xrightarrow{s'} \mathcal{N}_X(-H) \xrightarrow{s' \wedge} \mathcal{J}_{Z_{s'}}(3H - \Sigma) \longrightarrow 0.$$

From this sequence, it follows that the global section $es = e\sigma s' \in H^0(\mathcal{N}_X(-H))$, which interests us, lies in the kernel of the homomorphism $H^0(\mathcal{N}_X(-H)) \rightarrow H^0(\mathcal{J}_{Z_{s'}}(3H - \Sigma))$. Furthermore, by the definition of $v \in H^0(\mathcal{O}_X(H))^*$, we have

$$es = \eta(v),$$

where η is the slanted arrow in the diagram (10.5). On the other hand, the composition

$$H^0(\mathcal{O}_X(H))^* \otimes \mathcal{O}_X \xrightarrow{\eta} \mathcal{N}_X(-H) \xrightarrow{s' \wedge} \mathcal{J}_{Z_{s'}}(3H - \Sigma) \xrightarrow{\sigma} \mathcal{J}_{Z_{s'}}(3H)$$

is given by the partial derivatives of a polynomial defining V_4 . More precisely, if F is a homogeneous polynomial defining V_4 , then the composition $\sigma \circ (s' \wedge \cdot) \circ \eta$ sends every $w \in H^0(\mathcal{O}_X(H))^*$ to $\partial_w(F)|_X$, the derivative of F in the direction of w restricted to X . Evaluating the composition on the vector v , we obtain

$$\partial_v(F)|_X = \sigma \cdot ((s' \wedge \cdot) \circ \eta)(v) = \sigma(s' \wedge \eta(v)) = \sigma(s' \wedge (es)) = \sigma s' \wedge (e\sigma s') = 0.$$

Since X is contained in no hypersurface of degree 3, the above implies that $\partial_v(F) = 0$ in $\text{Sym}^3(H^0(\mathcal{O}_X(H)))$. Equivalently, $F \in \text{Sym}^4(\ker(v))$, i.e., the quartic hypersurface V_4 is the cone in $\mathbb{P}(H^0(\mathcal{O}_X(H))^*)$ with vertex $[v]$ and base the quartic surface S defined by F in $\mathbb{P}(\ker(v)^*) \cong \mathbb{P}^3$. \square

In the course of the proof of the preceding lemma we introduced the divisorial part Σ of the section $s \in H^0(\mathcal{N}_X(-K_X - H))$ defined by V_4 . Geometrically, Σ is the 1-dimensional part of the singular locus of V_4 contained in X . Our next result shows

LEMMA 10.4. $\Sigma = 0$.

Proof of Lemma 10.4. Assume Σ is not zero. Then we claim that Σ is composed of (-1) -curves which are lines with respect to H . Indeed, let C be a reduced, irreducible component of Σ and let γ be a global section defining C . Then $s = s'\gamma$, with $s' \in H^0(\mathcal{N}_X(-K_X - H - C))$. This section gives the cohomology class $\delta_X(s') = \xi' \in H^1(\Theta_X(-K_X - H - C))$ which is related to ξ by the formula $\gamma\xi' = \xi$. In particular, ξ' is nonzero and gives rise to a nontrivial extension

$$0 \longrightarrow \mathcal{O}_X(-K_X - H - C) \longrightarrow \mathcal{T}_{\xi'} \longrightarrow \Omega_X \longrightarrow 0. \quad (10.6)$$

In contrast to the homomorphism induced by the extension (10.1), the homomorphism

$$H^0(\mathcal{T}_{\xi'}) \longrightarrow H^0(\Omega_X) \quad (10.7)$$

induced by the epimorphism in (10.6) fails to be an isomorphism, since otherwise the argument of the first part of the proof gives a decomposition $K_X = L' + E'$, with L' effective, while Lemma 2.3, 3), applied to $\xi' = \delta_X(s')$ yields $-(L' + C) \cdot H \geq 0$, an obvious contradiction.

The failure of the homomorphism (10.7) to be an isomorphism implies that $H^1(\mathcal{O}_X(-K_X - H - C)) \neq 0$. But from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-K_X - H - C) \longrightarrow \mathcal{O}_X(-K_X - H) \longrightarrow \mathcal{O}_C(-K_X - H) \longrightarrow 0$$

and Lemma 10.2, it follows

$$H^0(\mathcal{O}_C(-K_X - H)) \cong H^1(\mathcal{O}_X(-K_X - H - C)) \neq 0.$$

Hence $(K_X + H) \cdot C \leq 0$. In the proof of Lemma 10.2 such a curve C is identified as a (-1) -curves with $H \cdot C = 1$.

Next we wish to identify the configuration of the lines composing Σ inside the quartic cone V_4 . We begin by observing that no line in Σ can be a ruling of the cone, since otherwise such a ruling L must be contained in the singular locus of V_4 and hence connect the vertex

$[v]$ to a singular point of a base S of the cone; but then the linear system $|H - L|$ must have L in its base locus which is absurd since $|H - L|$ is base point free—the linear system $|H - L|$ corresponds to the projection of X from the line L and this is a morphism $X \rightarrow \mathbb{P}^2$.

Once we know that the lines composing Σ are not rulings of V_4 , we deduce that their projection from the vertex $[v]$ are lines contained in the singular locus of a base S of the cone. Thus each line C of Σ is contained in the plane P_C spanned by C and $[v]$. The plane P_C is contained in the singular locus of V_4 . In addition, since the lines composing Σ are disjoint²⁰, C is the only component of Σ contained in P_C . From this it follows that the one dimensional part of $X \cap P_C$ is the line C . This means that the pencil of hyperplanes $\{V_t\}_{t \in \mathbb{P}^1}$ in \mathbb{P}^4 cutting out the plane P_C , intersects X along reducible divisors $H_t = V_t \cdot X$ containing the line C with multiplicity at least 2. From this it follows that the plane P_C is the embedded tangent plane of X for every $x \in C$. But this contradicts the fact that the Gauss map of X is finite, see [36]. \square

The last lemma implies that the quartic surface S in Lemma 10.3 has at most isolated singularities and the hypersurface V_4 is the cone over S with vertex $[v]$. Our surface X is a smooth divisor in the cone V_4 and the projection from the vertex $[v]$ defines a morphism

$$p_v : X \longrightarrow S$$

onto a normal quartic surface $S \subset \mathbb{P}^3$. In particular, the degree of p_v is given by the formula

$$\deg(p_v) = \begin{cases} \frac{d}{4}, & \text{if } [v] \notin X \\ \frac{d-1}{4}, & \text{if } [v] \in X. \end{cases}$$

Since the singular locus of V_4 is the union of its rulings joining $[v]$ to the points of $\text{Sing}(S)$, the singular locus of S , and by Lemma 10.4 none of these is contained in X , we deduce that the zero-locus $Z_0 = (s = 0)$ of the section $s \in H^0(\mathcal{N}_X(-K_X - H))$ defined by V_4 is 0-dimensional of degree

$$\deg(Z_0) = \begin{cases} \deg(\text{Sing}(S)) \cdot \frac{d}{4}, & \text{if } [v] \notin X, \\ \deg(\text{Sing}(S)) \cdot \frac{d-1}{4} + 1, & \text{if } [v] \in X. \end{cases}$$

The above formulas and the well known fact that $\deg(\text{Sing}(S)) \leq 16$ imply

$$\deg(Z_0) \leq 4d. \tag{10.8}$$

Next we relate the normal sequence of X with the extension sequence (10.1) defined by $\xi = \delta_X(s) \in H^1(\Theta_X(-K_X - H))$. For this we write the section s as a morphism of sheaves

$$\mathcal{O}_X(K_X + H) \xrightarrow{s} \mathcal{N}_X. \tag{10.9}$$

Applying $\text{Ext}^1(\bullet, \Theta_X)$, we have

$$\text{Ext}^1(\mathcal{N}_X, \Theta_X) \xrightarrow{s} \text{Ext}^1(\mathcal{O}_X(K_X + H), \Theta_X). \tag{10.10}$$

In particular, the extension class $\mathfrak{n} \in \text{Ext}^1(\mathcal{N}_X, \Theta_X)$ corresponding to the normal sequence

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow \mathcal{N}_X \longrightarrow 0$$

²⁰If there are two intersecting lines C and C' in Σ , then $(C + C')^2 = 0$ and this contradicts the assumption that X is of Albanese dimension 2.

goes under the homomorphism in (10.10) to the extension class $\mathbf{n} \cdot s$. But the coboundary map $\delta_X : H^0(\mathcal{N}_X(-K_X - H)) \rightarrow H^1(\Theta_X(-K_X - H))$ is precisely the cup-product with \mathbf{n} . Hence $\mathbf{n} \cdot s = \delta_X(s) = \xi$. This means that the morphism in (10.9) extends to a morphism of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_X & \longrightarrow & \mathcal{T}_\xi^* & \longrightarrow & \mathcal{O}_X(K_X + H) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow s \\ 0 & \longrightarrow & \Theta_X & \longrightarrow & \Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X & \longrightarrow & \mathcal{N}_X \longrightarrow 0 \end{array}$$

where the sequence on the top is the dual of (10.1). This can be completed to the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta_X & \longrightarrow & \mathcal{T}_\xi^* & \longrightarrow & \mathcal{O}_X(K_X + H) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow s \\ 0 & \longrightarrow & \Theta_X & \longrightarrow & \Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X & \longrightarrow & \mathcal{N}_X \longrightarrow 0 \\ & & & & \downarrow & & \downarrow s \wedge \\ & & & & \mathcal{J}_{Z_0}(4H) & \equiv & \mathcal{J}_{Z_0}(4H) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \quad (10.11)$$

This diagram will enable us to control the subscheme Z in the destabilizing sequence (10.4) of \mathcal{T}_ξ .

LEMMA 10.5. *The subscheme Z is either $[v]$ or empty. In particular, $\deg(Z) \leq 1$.*

Proof of Lemma 10.5. Dualizing (10.4) and tensoring it with $\mathcal{O}_X(-H)$, we obtain a section $t \in H^0(\mathcal{T}_\xi^*(-H))$ whose zero locus is Z . On the other hand the middle column of the diagram (10.11) tensored with $\mathcal{O}_X(-H)$ tells us that t can be identified with the section $\tau \in H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-H))$ corresponding to the vector v defining the vertex $[v]$ of V_4 . In particular, the scheme $Z_\tau = (\tau = 0) \supset Z$. To understand Z_τ we consider the contraction with τ

$$\lrcorner \tau : H^0(\Omega_{\mathbb{P}^4}(2) \otimes \mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_X(H)).$$

This implies that the image of $\lrcorner \tau$ is contained in $H^0(\mathcal{J}_Z(H))$. But

$$H^0(\Omega_{\mathbb{P}^4}(2) \otimes \mathcal{O}_X) \cong \wedge^2 H^0(\mathcal{O}_X(H))$$

and under the identification of τ with v , the above contraction becomes simply the contraction with $v \in H^0(\mathcal{O}_X(H))^*$,

$$\lrcorner v : \wedge^2 H^0(\mathcal{O}_X(H)) \longrightarrow H^0(\mathcal{O}_X(H)).$$

This implies

$$\text{im}(\lrcorner \tau) = \text{im}(\lrcorner v) = v^\perp,$$

i.e., $\text{im}(\lrcorner \tau)$ is identified with the space of hyperplanes in \mathbb{P}^4 passing through $[v]$. Hence, $Z \subset Z_\tau$ is either \emptyset or $[v]$. \square

We now have everything to rule out the existence of X . Indeed, from the destabilizing sequence of \mathcal{T}_ξ we have

$$\deg(Z) = c_2(\mathcal{T}_\xi) = c_2(\mathcal{T}_\xi^*).$$

This and the middle column in (10.11) give

$$10d = c_2(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X) = c_2(\mathcal{T}_\xi^*) + \deg(Z_0) + 4d = \deg(Z) + \deg(Z_0) + 4d.$$

From this and Lemma 10.5 it follows

$$\deg(Z_0) \geq 6d - 1.$$

Putting together this inequality and the upper bound for $\deg(Z_0)$ in (10.8), we obtain $2d \leq 1$ which is clearly impossible. \square

10.2. The surfaces $X \subset \mathbb{P}^4$ with $m_X = 5$ and of Albanese dimension 2

We now turn to the consideration of surfaces $X \subset \mathbb{P}^4$ of Albanese dimension 2 and $m_X = 5$. To apply our method in this case we need the additional assumption of X being minimal. With this in mind we proceed as in the case $m_X = 4$. Namely, let V_5 be a quintic hypersurface containing X and let s be a nonzero global section of $\mathcal{N}_X(-K_X)$ defined by V_5 . That section is used to obtain the cohomology class $\xi = \delta_X(s)$, which by Lemma 2.2 is nonzero. The class ξ is interpreted as a nontrivial extension

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0 \quad (10.12)$$

which we use to gain an insight into the geometry of X and V_5 .

LEMMA 10.6. *X is either of general type or an abelian surface of degree $d = 10$.*

Proof. The assumption that X is of Albanese dimension 2 implies that K_X is effective. The minimality of X ensures that K_X is nef. Hence either $K_X^2 > 0$ and X is of general type, or $K_X^2 = 0$. In the latter case, we deduce that $K_X = 0$, and by the Enriques-Kodaira classification, X is an abelian surface. From this and the double point formula, we obtain that X is of degree $d = 10$. \square

Abelian surfaces of degree 10 are of course Horrocks-Mumford surfaces and it is well-known that they are contained in a quintic hypersurface and not in one of a smaller degree. From now on we assume that X is of general type. This assumption implies that the homomorphism

$$H^0(\mathcal{T}_\xi) \longrightarrow H^0(\Omega_X)$$

induced by the epimorphism in (10.12) is an isomorphism. This allows us to define the saturation \mathcal{G} of the subsheaf of \mathcal{T}_ξ generated by its global sections.

LEMMA 10.7. *The following possibilities may arise:*

- 1) *The rank of \mathcal{G} is 3 and then $\mathcal{G} = \mathcal{T}_\xi \cong \mathcal{O}_X^{\oplus 3}$. In particular, the irregularity $q = 3$, Ω_X is generated by its global sections and $K_X^2 = c_2$.*
- 2) *The rank of \mathcal{G} is 2.*

Proof. From the definition of \mathcal{G} and the assumption that X is of Albanese dimension 2, it follows that the rank of \mathcal{G} is at least 2.

If $\text{rank}(\mathcal{G}) = 3$, then one can choose a subspace $V \subset H^0(\mathcal{G})$ of dimension 3 so that the evaluation morphism $V \otimes \mathcal{O}_X \rightarrow \mathcal{G}$ is generically an isomorphism. This, followed by the inclusion $\mathcal{G} \hookrightarrow \mathcal{T}_\xi$, gives a morphism

$$V \otimes \mathcal{O}_X \longrightarrow \mathcal{T}_\xi$$

which is generically an isomorphism. Since $\det(\mathcal{T}_\xi) = \mathcal{O}_X$, the above morphism must be an isomorphism and we obtain the identifications

$$V \otimes \mathcal{O}_X \cong \mathcal{G} = \mathcal{T}_\xi.$$

This implies

$$H^0(\Omega_X) \cong H^0(\mathcal{G}) \cong H^0(V \otimes \mathcal{O}_X) = V \cong \mathbb{C}^3.$$

Hence $q = 3$ and the exact sequence (10.12) takes the form

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow 0.$$

In particular, the above sequence implies that the cotangent bundle Ω_X is generated by its global sections and the Chern numbers of X are subject to $K_X^2 = c_2$. \square

We now investigate the case $\text{rank}(\mathcal{G}) = 2$.

LEMMA 10.8. *If $\text{rank}(\mathcal{G}) = 2$, then $q = h^0(\mathcal{G}) = 2$ or 3. Furthermore, if $q = 3$, then the following possibilities may arise:*

- 1) $\det(\mathcal{G}) = \mathcal{O}_X(H)$.
- 2) $h^0(\det(\mathcal{G})) = 2$ and X admits a fibration $p : X \rightarrow B$ onto a smooth curve B of genus $g_B = 2$ such that

$$\det(\mathcal{G}) = \mathcal{O}_X(p^*K_B + R),$$

where K_B is the canonical divisor of B and R is the fixed part of the linear system $|\det(\mathcal{G})|$.

Proof. The condition $\text{rank}(\mathcal{G}) = 2$ allows us to apply Lemma 2.3 and deduce:

- i) the decomposition of the canonical divisor

$$K_X = L + E,$$

where $L = c_1(\mathcal{G})$ and E is an effective nonzero divisor supported on the cokernel of the morphism $\varphi : \mathcal{G} \rightarrow \Omega_X$,

- ii) a nonzero global section $\tau \in H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L))$.

To analyse the situation further, we take a nonzero global section g of \mathcal{G} and write the exact sequence

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow \mathcal{G} \longrightarrow \mathcal{J}_A(L - F) \longrightarrow 0, \quad (10.13)$$

associated to g . In this sequence, F is the divisorial part of the scheme $Z_g = (g = 0)$, A is the 0-dimensional subscheme obtained after dividing g by a section of $\mathcal{O}_X(F)$ corresponding

to F , and \mathcal{J}_A is the ideal sheaf of A . The assumption that the Albanese dimension of X is 2 implies that the homomorphism

$$H^0(\mathcal{G}) \longrightarrow H^0(\mathcal{J}_A(L - F))$$

induced by the epimorphism in (10.13) is nonzero. In particular,

$$h^0(\mathcal{O}_X(L - F)) \geq h^0(\mathcal{G}) - h^0(\mathcal{O}_X(F)) = q - h^0(\mathcal{O}_X(F)) \geq 1$$

and we have the estimate

$$\begin{aligned} h^0(\mathcal{O}_X(L)) &\geq h^0(\mathcal{O}_X(F)) + h^0(\mathcal{O}_X(L - F)) - 1 \\ &\geq h^0(\mathcal{O}_X(F)) + q - h^0(\mathcal{O}_X(F)) - 1 = q - 1, \end{aligned} \quad (10.14)$$

implying $h^0(\mathcal{O}_X(L)) \geq 2$ if $q \geq 3$.

From now on we assume $q \geq 3$ and write

$$L = M + R, \quad (10.15)$$

where $|M|$ and R are the moving and the fixed part of $|L|$ respectively. We divide our considerations according to the dimension $h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_X(M))$.

Case $h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_X(M)) \geq 3$. The nonzero global section $\tau \in H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L))$, see the beginning of the proof, gives rise to a nonzero global section of $\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-M)$. Since M is nef and big²¹, the Euler sequence for $\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X$ tensored with $\mathcal{O}_X(-M)$ implies $H^0(\mathcal{O}_X(H - M)) \neq 0$.

We claim that $M = H$. Indeed, assuming $\Gamma = H - M \neq 0$, we have

$$H^0(\mathcal{O}_X(H - \Gamma)) = H^0(\mathcal{O}_X(M)) \cong H^0(\mathcal{O}_X(L)).$$

This implies $h^0(\mathcal{O}_X(H - \Gamma)) = h^0(\mathcal{O}_X(L)) \geq 3$. Hence Γ must be a line and the last inequality must be an equality. Since

$$H = M + \Gamma = L - R + \Gamma = K_X + \Gamma - E - R,$$

taking the intersection with Γ on both sides, we deduce

$$1 = H \cdot \Gamma = (K_X + \Gamma) \cdot \Gamma - (E + R) \cdot \Gamma = -2 - (E + R) \cdot \Gamma.$$

From this, it follows

$$R \cdot \Gamma = -3 - E \cdot \Gamma.$$

In addition, if Γ is not in E , then $E \cdot \Gamma \geq 0$, and if Γ is an irreducible component of E , then $E \cdot \Gamma \geq -2$, see [26]. This and the above identity imply $R \cdot \Gamma \leq -1$. Hence Γ is a component of R and we can rewrite the relation (10.15) as follows

$$L = M + R = M + \Gamma + R' = H + R',$$

where R' is an effective divisor. But this gives the contradiction

$$3 = h^0(\mathcal{O}_X(L)) \geq h^0(\mathcal{O}_X(H)) = 5.$$

²¹ $|M|$ is not composed of a pencil, since otherwise the image of the Albanese map is 1-dimensional.

Once we know that $M = H$, the identity (10.15) reads $L = H + R$ and we claim that $R = 0$. Indeed, if this is not the case, we take C , a reduced, irreducible component of R and use the fact that $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-H - C)) \neq 0$. This together with the Euler sequence implies $H^1(\mathcal{O}_X(-H - C)) \neq 0$. But from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

tensoring with $\mathcal{O}_X(-H)$, we see that $H^1(\mathcal{O}_X(-H - C)) \cong H^0(\mathcal{O}_C(-H)) = 0$.

Next we show that $q = 3$. For this we go back to the exact sequence (10.13) which now takes the form

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow \mathcal{G} \longrightarrow \mathcal{J}_A(H - F) \longrightarrow 0. \quad (10.16)$$

Furthermore, we may assume that there are two linearly independent global sections g and g' of \mathcal{G} which are proportional, *i.e.*, the exterior product $g \wedge g'$ is zero, viewed as a section of $\det(\mathcal{G})$ (otherwise we are done by [26, Lemma 5.4]). With such a choice of $g \in H^0(\mathcal{G})$ in constructing the exact sequence (10.16), we obtain that the line bundle $\mathcal{O}_X(F)$ in that sequence has $h^0(\mathcal{O}_X(F)) \geq 2$. From the isomorphism $H^0(\mathcal{G}) \cong H^0(\Omega_X)$ it also follows that the sections g, g' correspond to two linearly independent holomorphic 1-forms, call them ω, ω' , subject to $\omega \wedge \omega' = 0$ as a section of $\mathcal{O}_X(K_X)$. By Castelnuovo-de Franchis theorem, it follows that X admits a morphism

$$p : X \longrightarrow B \quad (10.17)$$

onto a smooth curve B of genus $g_B \geq 2$ such that $\omega = p^*(\eta)$ and $\omega' = p^*(\eta')$, where $\eta, \eta' \in H^0(\mathcal{O}_B(K_B))$. Hence

$$p^*(\mathcal{O}_B(K_B)) = \mathcal{O}_X(F). \quad (10.18)$$

Furthermore, from the assumption on the Albanese dimension of X we know that $q > g_B$. Hence

$$H^0(\mathcal{J}_A(H - F)) \supset \text{coker}(H^0(\mathcal{O}_X(F)) \longrightarrow H^0(\mathcal{G})) \neq 0.$$

In particular, $\Gamma = H - F$ is an effective nonzero divisor. From this it follows that

$$h^0(\mathcal{O}_X(F)) = h^0(\mathcal{O}_X(H - \Gamma)) \leq 3,$$

with the equality holding if and only if Γ is a line in \mathbb{P}^4 . We claim that the equality is impossible. Indeed, if Γ is a line, then

$$|F| = |H - \Gamma|$$

is base point free and hence, by (10.18), the divisor F is composed of the fibres of the morphism p in (10.17). But Γ is a rational curve and hence, must be contained in a fibre of that morphism. Then

$$0 = F \cdot \Gamma = (H - \Gamma) \cdot \Gamma = 1 - \Gamma^2.$$

Since $\Gamma^2 < 0$, the above identity is an obvious contradiction.

Thus $h^0(\mathcal{O}_X(F)) \leq 2$. This and the hypothesis $h^0(\mathcal{O}_X(F)) \geq 2$ imply

$$h^0(\mathcal{O}_X(F)) = 2.$$

which, together with (10.18) leads to $g_B = 2$.

Next we claim that $h^0(\mathcal{O}_X(H - F)) = 1$. Indeed, assume $h^0(\mathcal{O}_X(H - F)) \geq 2$. This means that the divisors of the linear system $|F| = |p^*(K_B)|$ are contained in a plane in \mathbb{P}^4 .

But a general divisor of $|F|$ contains two disjoint irreducible curves and this can not happen for plane curves.

The case $h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_X(M)) = 2$. In this case the estimate (10.14) tells us that $q \leq 3$ and hence, $q = 3$, in view of the assumption $q \geq 3$. Furthermore, the natural homomorphism

$$\wedge^2 H^0(\mathcal{G}) \longrightarrow H^0(\det(\mathcal{G})) = H^0(\mathcal{O}_X(L))$$

has a nontrivial kernel and this implies, as in the previous case, that the sheaf $\mathcal{O}_X(F)$ in (10.13) satisfies $h^0(\mathcal{O}_X(F)) \geq 2$. By the hypothesis on the Albanese dimension, the sheaf $\mathcal{J}_A(L-F)$ in that exact sequence must have $h^0(\mathcal{J}_A(L-F)) \geq 1$. This and the first inequality in (10.14) tells us that $h^0(\mathcal{O}_X(F)) \leq h^0(\mathcal{O}_X(L)) = 2$. Hence $h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_X(F)) = 2$ and $h^0(\mathcal{O}(L-F)) = 1$.

From the isomorphism $H^0(\mathcal{G}) \cong H^0(\Omega_X)$ we also conclude that $H^0(\mathcal{O}_X(F))$ defines a two dimensional subspace of $H^0(\Omega_X)$ contained in the kernel of the natural homomorphism

$$\wedge^2 H^0(\Omega_X) \longrightarrow H^0(\det(\Omega_X)) = H^0(\mathcal{O}_X(K_X)).$$

By Castelnuovo–de Franchis theorem, this means that X admits a morphism $p : X \rightarrow B$ onto a smooth curve B of genus $g_B \geq 2$. This together with the hypothesis on the Albanese dimension of X imply

$$2 \leq g_B < q = 3.$$

Hence $g_B = 2$ and $\mathcal{O}_X(F) = p^*(\mathcal{O}_B(K_B))$. In addition, from $h^0(\mathcal{O}_X(F)) = h^0(\mathcal{O}_X(L)) = 2$, we deduce the formula

$$L = p^*K_B + R,$$

where R is the fixed part of $|L|$. □

We summarize the above discussion in the following statement.

THEOREM 10.9. *If $X \subset \mathbb{P}^4$ is a minimal surface with $m_X = 5$ and of Albanese dimension 2, then its irregularity $q(X) = 2$ or 3. Furthermore, if $q(X) = 3$, then one of the following possibilities may occur:*

- 1) *The cotangent bundle Ω_X is generated by its global sections and $K_X^2 = c_2$,*
- 2) *The canonical divisor K_X admits the decomposition $K_X = L + E$, with L and E effective nonzero divisors. The decomposition is subject to the following properties:*
 - a) *The divisor L is either equal to H , or X admits a fibration $p : X \rightarrow B$ over a smooth curve B of genus 2 and $L = p^*(K_B) + R$, where K_B is the canonical divisor of B and R is the fixed part of the linear system $|L|$.*
 - b) *$L \cdot H \leq H^2 = d$ and $K_X^2 - c_2 \leq \frac{2}{3} L^2$.*

Proof. The assertions 1) and 2), a), are Lemma 10.7 and Lemma 10.8 respectively. The assertion b) is Lemma 2.3, 3) and c) is as in Theorem 4.4, ii). □

A. THE PROJECTIVE BUNDLE $\mathbb{P}(\mathcal{N}_X(-3H))$ AND THE EMBEDDING $X \subset \mathbb{P}^4$

In this appendix we return to an elliptic scroll X of degree 5 in \mathbb{P}^4 . In Theorem 8.1 we established an isomorphism between the space of global sections of $\mathcal{N}_X^*(3H)$ and the

space of cubic hypersurfaces $I_X(3)$ containing X , see Section 8 for notation. The subscheme $Z_s = (s = 0)$ of zeros of a nonzero global section s of $\mathcal{N}_X^*(3H)$ is identified with the scheme-theoretic intersection of X with the singular locus $\text{Sing}(V_3(s))$ of the cubic hypersurface $V_3(s)$ corresponding to s under the isomorphism

$$H^0(\mathcal{N}_X^*(3H)) \cong I_X(3). \quad (\text{A.1})$$

This, in particular, allows for a purely geometric way to recover X from the space $I_X(3)$ —a geometric counterpart of a well known algebraic fact that the homogeneous ideal of X is generated in degree 3.

More conceptually, the isomorphism (A.1) suggests a sort of ‘duality’ between $\mathcal{N}_X^*(3H)$ and X embedded in \mathbb{P}^4 by $\mathcal{O}_X(H)$. This is the main theme of this section. Of course, one aspect of the above mentioned duality is well-known—the famous quadro-cubic transformation of Cremona relating a normal elliptic quintic curve in \mathbb{P}^4 with an elliptic scroll (in another copy of \mathbb{P}^4) see [4, 5]. So we do not claim any novelty in the results exposed here. However, placing the vector bundle $\mathcal{N}_X^*(3H)$ in the center of the study, revisiting various aspects of the geometry of the scroll X and of the Segre cubics containing it via the properties of $\mathcal{N}_X^*(3H)$, seem to be new and fruitful.

Set $\mathcal{E} = \mathcal{N}_X(-3H)$. The projectivization $Y = \mathbb{P}(\mathcal{E})$ with the structure projection

$$p : Y = \mathbb{P}(\mathcal{E}) \longrightarrow X \quad (\text{A.2})$$

is equipped with the line bundle $\mathcal{O}_Y(1)$ chosen so that the direct image

$$p_*\mathcal{O}_Y(1) = \mathcal{E}^* = \mathcal{N}_X^*(3H).$$

By Lemma 8.2, the vector bundle \mathcal{E}^* is globally generated. Hence $\mathcal{O}_Y(1)$ is globally generated and defines a morphism

$$\varphi : Y = \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(H^0(\mathcal{E}^*)^*) \quad (\text{A.3})$$

where the target space is, in view of Theorem 8.1, 2), a 4-dimensional projective space. We let

$$V := H^0(\mathcal{O}_X(H)) \quad \text{and} \quad W := H^0(\mathcal{E}^*) = H^0(\mathcal{N}_X^*(3H))$$

and want to keep a clear distinction between the following two geometric incarnations of the projective space \mathbb{P}^4 :

- the projective space $\mathbb{P}(V^*)$, where X lives,
- the projective space $\mathbb{P}(W^*)$, the target of the morphism φ defined in (A.3).

One of our goals here is to describe how to go between these two spaces.

A.1. On the geometry of the scroll X

To describe the geometry of the morphism φ (resp. of the embedding $X \subset \mathbb{P}(V^*)$) we recall that X is a \mathbb{P}^1 -bundle over an elliptic curve which will be denoted by E and we let

$$\pi : X \longrightarrow E \quad (\text{A.4})$$

be the structure projection. It is well known that X can be identified with $\text{Sym}^2(E)$, the second symmetric power of E . Then the structure projection becomes the Abel-Jacobi map

which takes a subscheme $D \subset E$ of degree 2, viewed as a point of $\text{Sym}^2(E)$, to (the isomorphism class of) the line bundle $\mathcal{O}_E(D)$, viewed as a point of E . We are making here a (non-canonical) identification of E with $\text{Jac}_2(E)$, the variety of isomorphism classes of line bundles of degree 2 on E . From this it follows that X admits the obvious double covering

$$\tau : E \times E \longrightarrow \text{Sym}^2(E) = X \quad (\text{A.5})$$

which sends a point $(e, e') \in E \times E$ to $\tau(e, e')$, the subscheme of E of degree 2 supported on e and e' . This equips X with a distinguished family of curves

$$\Gamma_e := \tau(\nu_1^{-1}(e)) = \tau(\nu_2^{-1}(e)), \quad (\text{A.6})$$

where $e \in E$ and ν_i denotes the projection of $E \times E$ on the i -th factor, for $i = 1, 2$. These curves will play an important role in the sequel, so we mention some of their properties, immediate consequences of the definition (A.6).

- i) For every $e \in E$, the curve Γ_e is a section of the projection π in (A.4).
 - ii) $\Gamma_e \cdot \Gamma_{e'} = 1$, for every $e, e' \in E$.
 - iii) $h^0(\mathcal{O}_X(\Gamma_e)) = 1$.
- (A.7)

The curves $\{\Gamma_e\}_{e \in E}$ are used to define embeddings of X into \mathbb{P}^4 .

PROPOSITION A.1. *Let $\mathcal{O}_E(D)$ be a line bundle of degree 2 on E . Then for every point $e \in E$ the line bundle $\mathcal{O}_X(\Gamma_e + \pi^*(D))$ is very ample and it defines an embedding of X into \mathbb{P}^4 as a scroll of degree 5.*

Proof. Let l be the class of a fibre of $\pi : X \rightarrow E$ in the Néron-Severi group of X . The canonical divisor has the form $K_X = -2\Gamma_e + l$ and we set

$$H := \Gamma_e + \pi^*(D) = K_X + (H - K_X), \quad (\text{A.8})$$

where $H - K_X = 3\Gamma_e + l$ is nef and big. The very ampleness of H now follows easily from [29, Theorem 1, 2)].

From the formula (A.8), we see that $h^i(\mathcal{O}_X(\Gamma_e + \pi^*(D))) = 0$, for $i = 1, 2$. Hence by Riemann-Roch we obtain

$$h^0(\mathcal{O}_X(\Gamma_e + \pi^*(D))) = \chi(\mathcal{O}_X(\Gamma_e + \pi^*(D))) = \frac{1}{2} ((\Gamma_e + 2l)^2 - (\Gamma_e + 2l) \cdot (-2\Gamma_e + l)) = 5.$$

Thus $\mathcal{O}_X(H)$ embeds X into \mathbb{P}^4 . Moreover, since $H \cdot l = (\Gamma_e + 2l) \cdot l = 1$, it follows that $\mathcal{O}_X(H)$ embeds X as a scroll of degree

$$H^2 = (\Gamma_e + 2l)^2 = \Gamma_e^2 + 4\Gamma_e \cdot l = 1 + 4 = 5.$$

□

From now on we fix a point $o \in E$ and a line bundle $\mathcal{O}_E(D)$ of degree 2 on E . According to the previous proposition, this gives an embedding of X into $\mathbb{P}(V^*)$ defined by $\mathcal{O}_X(H)$, where $H := \Gamma_o + \pi^*(D)$ and $V = H^0(\mathcal{O}_X(H))$.

PROPOSITION A.2. *Under the embedding of X defined by $\mathcal{O}_X(H)$, the curves $\{\Gamma_e\}_{e \in E}$ are embedded as plane curves of degree 3. Conversely, a plane cubic curve contained in X is one of the curves of the family $\{\Gamma_e\}_{e \in E}$.*

Proof. From $\mathcal{O}_X(H - \Gamma_e) = \mathcal{O}_X(\Gamma_o + \pi^*(D) - \Gamma_e) = \pi^*\mathcal{O}_E(D + (o - e))$ it follows that

$$h^0(\mathcal{O}_X(H - \Gamma_e)) = h^0(\pi^*\mathcal{O}_E(D + (o - e))) = h^0(\mathcal{O}_E(D + (o - e))) = 2.$$

This means that under the embedding by $\mathcal{O}_X(H)$, the image of Γ_e is contained in a plane. Its degree is computed by the intersection number

$$H \cdot \Gamma_e = (\Gamma_o + \pi^*(D)) \cdot \Gamma_e = 1 + 2 = 3,$$

where the properties *i*) and *ii*) in (A.7) are used.

Conversely, let C be a plane cubic contained in X . Then

$$h^0(\mathcal{O}_X(H - C)) = 2 \quad \text{and} \quad H \cdot (H - C) = 2.$$

Hence, the effective divisor $H - C$ is composed of rational curves. Since the only rational curves on X are the rulings (the fibres of π) of X , we deduce the linear equivalence

$$H - C \sim \pi^*(D'),$$

for some effective divisor D' of degree 2 on E . Hence

$$C \sim H - \pi^*(D') = \Gamma_o + \pi^*(D - D') = \Gamma_e,$$

where $e = o + e'$ and e' is the point of E corresponding to $\mathcal{O}_E(D - D')$ under the identification $E = \text{Jac}_0(E)$. Since $h^0(\mathcal{O}_X(\Gamma_e)) = 1$, see the property *iii*) in (A.7), it follows that $C = \Gamma_e$. \square

Remark A.3. Viewing X as a subvariety of $\mathbb{P}(V^*)$, one obtains the family $\{\Gamma_e\}_{e \in E}$ as follows: consider two rulings l_a and $l_{a'}$ of X lying over the points a and a' of E and consider the hyperplane $V_{a,a'}$ in $\mathbb{P}(V^*)$ spanned by the rulings l_a and $l_{a'}$. The hyperplane section $H_{a,a'} = V_{a,a'} \cap X$ is a reducible divisor in $|H|$ of the form

$$H_{a,a'} = C + l_a + l_{a'},$$

where C is the component of $H_{a,a'}$ of degree $H \cdot C = 3$, residual to $l_a + l_{a'}$. Hence it must be irreducible and a section of the projection $\pi : X \rightarrow E$. Therefore, C is a plane cubic contained in X and, in view of Proposition A.2, it must be a curve in $\{\Gamma_e\}_{e \in E}$. The first part of the proof of Proposition A.2 shows that every curve in $\{\Gamma_e\}_{e \in E}$ is obtained in this way.

The family of curves $\{\Gamma_e\}_{e \in E}$ gives rise to the family of planes $\{\Pi_e\}_{e \in E}$, where Π_e is the plane spanned by Γ_e . To understand the properties of this family of planes we have the following.

PROPOSITION A.4. *Let $\mathcal{F} := \nu_{1*}(\tau^*\mathcal{O}_X(H))$. Then \mathcal{F} is a rank 3 vector bundle on E of degree 5, generated by its global sections with $h^0(\mathcal{F}) = 5$.*

Proof. By definition, $\text{rank}(\mathcal{F} \otimes \mathcal{O}_{E,e}) = h^0(\mathcal{O}_{\Gamma_e}(H)) = 3$ for every $e \in E$. Hence \mathcal{F} is locally free of rank 3. By Riemann-Roch

$$\deg(\mathcal{F}) = \chi(\mathcal{F}) = \chi(\nu_{1*}(\tau^*\mathcal{O}_X(H))) = \chi(\tau^*\mathcal{O}_X(H)) = H^2 = 5.$$

Furthermore, $H^0(\mathcal{F}) \cong H^0(\tau^*\mathcal{O}_X(H)) \cong H^0(\mathcal{O}_X(H))$. Hence $h^0(\mathcal{F}) = h^0(\mathcal{O}_X(H)) = 5$.

The global generation of \mathcal{F} follows from the identifications

$$H^0(\mathcal{F}) \cong H^0(\mathcal{O}_X(H)) \longrightarrow H^0(\mathcal{O}_{\Gamma_e}(H)) \cong H^0(\mathcal{F} \otimes \mathcal{O}_{E,e})$$

and the fact that, for every $e \in E$, the above restriction homomorphism is surjective. \square

Set $T := \mathbb{P}(\mathcal{F}^*)$ to be the projectivization of the dual of \mathcal{F} and $\rho : T \rightarrow E$ the structure projection. Let $\mathcal{O}_T(1)$ be chosen so that $\rho_*\mathcal{O}_T(1) = \mathcal{F}$. In view of Proposition A.4, the line bundle $\mathcal{O}_T(1)$ is generated by its global sections and hence gives a morphism

$$\psi : T \longrightarrow \mathbb{P}(V^*)$$

By definition, the fibres of T are mapped by ψ to the family of planes $\{\Pi_e\}_{e \in E}$. In particular, we obtain

PROPOSITION A.5. *The image T' of ψ is a hypersurface of degree 5 containing X in its singular locus.*

Proof. From the definition of ψ it follows

$$\deg(\psi) \deg(T') = c_1^3(\mathcal{O}_T(1)) = \deg(\mathcal{F}) = 5,$$

where the last equality comes from Proposition A.4. The above implies $\deg(\psi) = 1$ and $\deg(T') = 5$.

The hypersurface T' obviously contains X . To see that X is contained in its singular locus, we observe that the planes P_e and $P_{e'}$, for $e \neq e' \in E$, intersect at a single point. Indeed, otherwise the hyperplane spanned by $P_e \cup P_{e'}$ gives rise to a divisor $D \in |H|$ of the form

$$D = \Gamma_e + \Gamma_{e'} + R$$

where R is some effective divisor. The divisor D has a wrong degree on the rulings of X .

The point of intersection $P_e \cap P_{e'}$ is obviously a singular point of T' . Furthermore, this point is the point of intersection $\Gamma_e \cdot \Gamma_{e'}$ and hence belongs to X . Varying e and e' , the points $\Gamma_e \cdot \Gamma_{e'}$ form a Zariski dense open subset of X . Therefore, X is contained in the singular locus of T' . \square

Besides being the union of the family of planes $\{\Pi_e\}_{e \in E}$, the variety T' in Proposition A.5 can be also characterized as follows.

PROPOSITION A.6. *The variety T' in Proposition A.5 is the union of the proper trisecant lines of $X \subset \mathbb{P}(V^*)$.*

Proof. Let Π_e^\vee be the dual of the plane Π_e . Then the family of the dual planes $\{\Pi_e^\vee\}_{e \in E}$ gives rise to a family of proper trisecants of X in $\mathbb{P}(V^*)$. We need to check that every proper trisecant of X belongs to this family. Let l be a proper trisecant line of X , i.e., $Z_l := l \cdot X$ is a 0-dimensional subscheme of X of degree at least 3. Then

$$h^1(\mathcal{J}_{Z_l}(H)) = \deg(Z_l) - 2 \geq 3 - 2 = 1.$$

This and the Serre duality $\text{Ext}^1(\mathcal{J}_{Z_l}(H), \mathcal{O}_X(K_X)) \cong H^1(\mathcal{J}_{Z_l}(H))^* \neq 0$ gives rise to a non-trivial extension sequence

$$0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z_l}(H) \longrightarrow 0.$$

The sheaf \mathcal{E} in the middle of that sequence is torsion free with $\det(\mathcal{E}) = \mathcal{O}_X(H + K_X)$ and $h^0(\mathcal{E}) \geq h^0(\mathcal{J}_{Z_l}(H)) - h^1(\mathcal{O}_X(K_X)) = 3 - 1 = 2$. Hence the saturation of the subsheaf generated by global section of \mathcal{E} is a torsion free subsheaf of \mathcal{E} of rank one. Hence it has the

form $\mathcal{J}_{Z_1}(A)$, where A (resp. Z_1) is an effective divisor (resp. 0-dimensional subscheme) on X , and gives rise to the following destabilizing sequence of \mathcal{E}

$$0 \longrightarrow \mathcal{J}_{Z_1}(A) \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{Z_2}(K_X + H - A) \longrightarrow 0.$$

Combining this together with the defining extension sequence implies that there is an effective nonzero divisor $\Gamma \in |H - A|$ passing through Z_l . From $h^0(\mathcal{O}_X(H - \Gamma)) = h^0(\mathcal{O}_X(A)) \geq 2$ and the fact that Γ can not be a line, it follows that the equality must hold and therefore, Γ is a plane curve. Hence Γ is a plane cubic section of X and, in view of Proposition A.2, $\Gamma = \Gamma_e$, for some $e \in E$. This implies $Z_l = l \cdot \Gamma_e \subset \Pi_e$ and hence the line l correspond to a point in the dual plane Π_e^\vee . \square

A.2. From the morphism φ in (A.3) to $X \subset \mathbb{P}(V^*)$

The above discussion gives a clear geometric picture on the side of the embedding $X \subset \mathbb{P}(V^*)$. We now turn to the other side, the projectivization $\mathbb{P}(\mathcal{E})$ and the morphism

$$\varphi : Y = \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(W^*)$$

introduced in (A.3), where $W = H^0(\mathcal{E}^*)$.

PROPOSITION A.7. *For $\varphi : Y = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(W^*)$, the following statements hold.*

- 1) *The 3-fold Y contains a distinguished divisor $\Sigma \cong E \times E$ and*

$$p|_\Sigma : E \times E \cong \Sigma \longrightarrow X \cong \text{Sym}^2(E),$$

the restriction to Σ of the structure projection p in (A.2), is the double covering τ defined in (A.5).

- 2) *Let Δ_E denote the diagonal of $E \times E$. Then the image $\varphi(E \times E) = \varphi(\Delta_E)$ is an embedding of E as an elliptic normal curve of degree 5.*
- 3) *The image $Y_0 = \varphi(Y) \subset \mathbb{P}(W^*)$ is a quintic hypersurface which is the secant variety of the elliptic normal curve $\varphi(\Delta_E)$.*
- 4) *The composition $\pi \circ p : Y \rightarrow E$ is a fibration whose fibres $(\pi \circ p)^{-1}(e) = S_e$ are isomorphic to the Hirzebruch surface $\Sigma_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. For every $e \in E$, the restriction $\varphi|_{S_e} : S_e \rightarrow \mathbb{P}(W^*)$ is an embedding. The image $S'_e := \varphi(S_e)$ is a rational cubic scroll containing the elliptic curve $\varphi(\Delta_E)$. This curve is the image under φ of the intersection $\Sigma \cdot S_e$. The intersection, viewed as a divisor in S_e , has the form $2L_e + 3f$, where L_e is a unique section of S_e with $L_e^2 = -1$ and f is the class of a fibre of p . In particular, the image $L'_e := \varphi(L_e)$ is a line intersecting $\varphi(\Delta_E)$ transversely at a single point.*
- 5) *The projection $p : Y \rightarrow X$ admits a distinguished section $\gamma : X \rightarrow Y$. The image of the composition $\varphi \circ \gamma : X \rightarrow \mathbb{P}(W^*)$ is a singular scroll of degree 15 containing $\varphi(\Delta_E)$.*

For the convenience of the reader, the following diagram summarizes all the morphisms appearing in (the proof of) the proposition and we suggest to consult this diagram while

advancing through the proof.

$$\begin{array}{ccccc}
 \Delta_E & \xrightarrow{\mu = \varphi|_{\Delta_E}} & & & \\
 \downarrow & & & & \\
 E \times E & \xrightarrow{i} & \mathbb{P}(\mathcal{E}) = Y & \xrightarrow{\varphi} & \mathbb{P}(W^*) \\
 \downarrow \tau & \swarrow p & & & \\
 \mathbb{P}(V^*) & \longleftarrow & X & & \\
 \downarrow \pi & & & & \\
 E & & & &
 \end{array}$$

Proof. We begin by showing that the natural double covering

$$\tau : E \times E \longrightarrow \text{Sym}^2(E) \cong X$$

admits a lifting to $Y = \mathbb{P}(\mathcal{E})$, *i.e.*, that there is a morphism $i : E \times E \rightarrow Y$ which fits into the following commutative diagram

$$\begin{array}{ccc}
 & Y = \mathbb{P}(\mathcal{E}) & \\
 & \uparrow i & \downarrow p \\
 E \times E & \xrightarrow{\tau} & X
 \end{array} \tag{A.9}$$

It is well known that this amounts to having a line bundle \mathcal{M} on $E \times E$ together with a surjective morphism

$$\tau^*(\mathcal{E}^*) \longrightarrow \mathcal{M},$$

see *e.g.*, [16], Ch. II, Proposition 7.12. In order to construct \mathcal{M} and the morphism, we investigate the restriction of \mathcal{E} to the curves $\{\Gamma_e\}_{e \in E}$ in (A.6).

LEMMA A.8. *For every $e \in E$, the restriction of \mathcal{E} to Γ_e has the form*

$$\mathcal{E} \otimes_{\mathcal{O}_{\Gamma_e}} \mathcal{N}_X(-3H) \otimes_{\mathcal{O}_{\Gamma_e}} \mathcal{O}_{\Gamma_e} = \mathcal{O}_{\Gamma_e} \oplus \mathcal{O}_{\Gamma_e}(K_X - H).$$

Proof of Lemma A.8. Set $\Gamma = \Gamma_e$. In view of Proposition A.2, the curve Γ is embedded into $\mathbb{P}(V^*)$ as a plane curve of degree 3. Hence its normal bundle $\mathcal{N}_{\Gamma/\mathbb{P}(V^*)}$ in $\mathbb{P}(V^*)$ has the form

$$\mathcal{N}_{\Gamma/\mathbb{P}^4} \cong \mathcal{O}_{\Gamma}(H) \oplus \mathcal{O}_{\Gamma}(H) \oplus \mathcal{O}_{\Gamma}(3H).$$

Using this decomposition in the short exact sequence of normal bundles of the inclusions $\Gamma \subset X \subset \mathbb{P}(V^*)$, we obtain

$$0 \longrightarrow \mathcal{O}_{\Gamma}(\Gamma) \longrightarrow \mathcal{O}_{\Gamma}(H) \oplus \mathcal{O}_{\Gamma}(H) \oplus \mathcal{O}_{\Gamma}(3H) \longrightarrow \mathcal{N}_X \otimes \mathcal{O}_{\Gamma} \longrightarrow 0.$$

This gives a nonzero morphism $\mathcal{O}_{\Gamma}(3H) \rightarrow \mathcal{N}_X \otimes \mathcal{O}_{\Gamma}$ or, equivalently, a nonzero global section, call it η , of $\mathcal{N}_X(-3H) \otimes \mathcal{O}_{\Gamma}$. This together with the fact that the dual vector bundle $\mathcal{N}_X^*(3H) \otimes \mathcal{O}_{\Gamma}$ is globally generated, see Theorem 8.1, implies that η is nowhere vanishing and gives a splitting

$$\mathcal{E} \otimes \mathcal{O}_{\Gamma} = \mathcal{N}_X(-3H) \otimes \mathcal{O}_{\Gamma} = \mathcal{O}_{\Gamma} \oplus \mathcal{O}_{\Gamma}(K_X - H)$$

as asserted. \square

We now take the pullback $\tau^*(\mathcal{E})$ and recall that the curve Γ_e , for every $e \in E$, is the image under τ of the fibre $\nu_1^{-1}(e)$ of the projection $\nu_1 : E \times E \rightarrow E$. This and the decomposition of Lemma A.8 imply that the direct image $\nu_{1*}(\tau^*(\mathcal{E}))$ is a line bundle, call it \mathcal{L} . This can be interpreted as a nowhere vanishing section of $\nu_{1*}(\nu_1^*(\mathcal{L}^{-1}) \otimes \tau^*(\mathcal{E}))$ which in turn gives a nowhere vanishing global section of $\nu_1^*(\mathcal{L}^{-1}) \otimes \tau^*(\mathcal{E})$ or, equivalently, the inclusion of bundles

$$\nu_1^*(\mathcal{L}) \hookrightarrow \tau^*(\mathcal{E}).$$

Dualizing gives the sought after epimorphism

$$\tau^*(\mathcal{E}^*) \longrightarrow \nu_1^*(\mathcal{L}^{-1}) \longrightarrow 0 \quad (\text{A.10})$$

and hence the diagram (A.9). Furthermore, under the morphism $i : E \times E \rightarrow Y$ associated to the above epimorphism, the line bundle $\nu_1^*(\mathcal{L}^{-1})$ becomes the pullback $i^*(\mathcal{O}_Y(1))$. Hence the composition morphism

$$\varphi \circ i : E \times E \longrightarrow \mathbb{P}(W^*)$$

factors through the morphism

$$\mu : E \longrightarrow \mathbb{P}(W^*), \quad (\text{A.11})$$

defined by \mathcal{L}^{-1} . Since the projection ν_1 restricted to the diagonal Δ_E induces the isomorphism of Δ_E with the base E (of the first projection), we deduce

$$(\varphi \circ i)(E \times E) = (\varphi \circ i)(\Delta_E).$$

Geometrically, the morphism i in (A.9), corresponding to the epimorphism (A.10), maps the fibre $\nu_1^{-1}(e) \cong \Gamma_e$ to the section of the ruled surface $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\Gamma_e})$ defined by the trivial summand of the decomposition $\mathcal{E} \otimes \mathcal{O}_{\Gamma_e} = \mathcal{O}_{\Gamma_e} \oplus \mathcal{O}_{\Gamma_e}(K_X - H)$ described in Lemma A.8. Under the morphism φ that section is contracted to the point $\mu(e)$, while the ruled surface $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\Gamma_e})$ is mapped by φ to the cone over a normal quartic elliptic curve²² with vertex at $\mu(e)$. From this and the properties of $\mathcal{N}^*(3H)$ we can draw several conclusions.

- a) The image $\mu(E)$ of the morphism μ in (A.11) can not be contained in a plane. Indeed, otherwise a plane containing $\mu(E)$ gives rise to a pair of linearly independent global sections of $\mathcal{E}^* = \mathcal{N}^*(3H)$ which are proportional, but such sections must have 1-dimensional zero locus and this, according to Theorem 8.1, 3), is impossible.
- b) The degree of μ onto its image is 1. Indeed, otherwise take two general points x and x' of $\mu(E)$ and let e_1 and e_2 (resp. e'_1 and e'_2) be two distinct points lying in the preimage $\mu^{-1}(x)$ (resp. $\mu^{-1}(x')$); now take a plane P passing through the points x and x' chosen on $\mu(E)$. Then all rulings of the cones $\varphi(\mathbb{P}(\mathcal{E} \otimes \Gamma_{e_i}))$ and $\varphi(\mathbb{P}(\mathcal{E} \otimes \Gamma_{e'_i}))$, for $i = 1, 2$, intersect P . From the identification

$$W = H^0(\mathcal{N}^*(3H)) \cong H^0(\mathcal{O}_Y(1)),$$

we deduce that two hyperplanes in $\mathbb{P}(W^*)$ cutting out the plane P give rise to two linearly independent global sections, say s and s' , of $\mathcal{N}^*(3H)$ such that $\gamma = s \wedge s'$, viewed as a global section of $\det(\mathcal{N}^*(3H)) = \mathcal{O}_X(H - K_X)$. The section γ vanishes along the divisor

$$\Gamma_{e_1} + \Gamma_{e_2} + \Gamma_{e'_1} + \Gamma_{e'_2} = 4\Gamma_o$$

which is impossible, since the divisor $H - K_X - 4\Gamma_o = -\Gamma_o + l$ can not be numerically equivalent to an effective divisor.

²²The base of the cone is the image of Γ_e by $\mathcal{O}_{\Gamma_e}(H - K_X)$ which is a line bundle of degree $(H - K_X) \cdot \Gamma_e = 3 + 1 = 4$.

Let E' be a Zariski dense open subset of E , where μ is an embedding. Take two distinct points e_1 and e_2 on E' . The corresponding curves Γ_{e_1} and Γ_{e_2} intersect at a point, call it x . From the properties established above, we know that the fibre L_x of the projection $p : Y \rightarrow X$ is mapped by φ into the line joining $\mu(e_1)$ and $\mu(e_2)$. Hence the secant variety of $\mu(E')$ is contained in the image $Y_0 = \varphi(Y)$. This implies, in view of the irreducibility of Y_0 , that the curve $\mu(E)$ spans $\mathbb{P}(W^*)$. Furthermore, for $e \in E'$, the union of the secant lines of $\mu(E)$ passing through $\mu(e)$ must be the cone $\varphi(\mathbb{P}(\mathcal{E} \otimes \Gamma_e))$ with the vertex $\mu(e)$ and base a smooth elliptic curve of degree 4. Hence this base curve is the image of the projection of $\mu(E)$ from the point $\mu(e)$. This implies that μ must be an embedding, $\mu(E)$ is an elliptic normal curve of degree 5, and the secant variety of $\mu(E)$ is contained in Y_0 . Since both are irreducible they must coincide. The assertion that the degree of Y_0 is 5 can now be deduced either from the fact that the secant variety of an elliptic normal curve in \mathbb{P}^4 has degree 5 or from the direct computation

$$\deg(Y_0) = c_1^3(\mathcal{O}_Y(1)) = c_1^2(\mathcal{E}^*) - c_2(\mathcal{E}^*) = (H - K_X)^2 - 10 = 15 - 10 = 5.$$

The second equality uses the Chern invariants of $\mathcal{E}^* = \mathcal{N}_X^*(3H)$ computed in Theorem 8.1.

The only remaining statement of Proposition A.7, 1)-3), to prove is that $i : E \times E \rightarrow Y$ in (A.9) is an embedding. From the work we have already done, this is immediate, since the image of i is a bi-section with respect to the projection $p : Y \rightarrow X$ and it is an isomorphism outside the diagonal Δ_E . On the diagonal Δ_E , we know that the composition $\varphi \circ i$ is an embedding. Hence i is an embedding everywhere.

Turning to the part 4) of Proposition A.7, we observe that the fibre $(\pi \circ p)^{-1}(e) = S_e$ over $e \in E$ is $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{F_e})$, where $F_e = \pi^{-1}(e)$. This together with

$$\mathcal{E} \otimes \mathcal{O}_{F_e} = \mathcal{N}_X(-3H) \otimes \mathcal{O}_{F_e} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$

(see (8.3) for the last isomorphism) proves that S_e is isomorphic to the Hirzebruch surface Σ_1 . Furthermore, the line bundle $\mathcal{O}_Y(1)$ restricted to S_e is $\mathcal{O}_{S_e}(1) := \mathcal{O}_Y(1) \otimes \mathcal{O}_{S_e}$ which is very ample, since $p_*(\mathcal{O}_{S_e}(1)) = \mathcal{N}_X^*(3H) \otimes \mathcal{O}_{F_e} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ is obviously very ample. The morphism $\varphi|_{S_e}$ is defined by $\mathcal{O}_{S_e}(1)$ and hence it embeds S_e into $\mathbb{P}(W^*)$ as a scroll of degree $\deg(\mathcal{N}_X^*(3H) \otimes \mathcal{O}_{F_e}) = \deg(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) = 3$, i.e., the image $S'_e = \varphi(S_e)$ is a rational normal scroll of degree 3 in $\mathbb{P}(W^*)$. Setting L_e to be the minimal section of S_e , we deduce

$$L_e = c_1(\mathcal{O}_{S_e}(1)) - 2f,$$

where f is the class of a fibre of p . In particular, L_e is a (-1) -curve on S_e and its image $L'_e = \varphi(L_e)$ is a line in $\mathbb{P}(W^*)$. The rulings of S'_e cut out on the elliptic curve $\varphi(\Delta_E)$ the pencil of degree 2 corresponding to the fibre $F_e \subset X \cong \text{Sym}^2(E)$. Hence $\varphi(\Delta_E)$ is a divisor on S'_e of the form $2L'_e + bl$ for some integer b , and where l , the image of f , is the class of a ruling of S'_e . The integer is determined from the equation

$$5 = \deg(\varphi(\Delta_E)) = c_1(\mathcal{O}_{S'_e}(1)) \cdot (2L'_e + bl) = 2 + b.$$

Hence $\varphi(\Delta_E) = 2L'_e + 3l$. From this it follows that

$$L'_e \cdot \varphi(\Delta_E) = L'_e \cdot (2L'_e + 3l) = -2 + 3 = 1.$$

Thus L'_e intersects $\varphi(\Delta_E)$ transversely at a single point.

The minimal section L_e of S_e is a distinguished section of p over the fibre $F_e = \pi^{-1}(e)$. Varying e in E gives rise to the section γ of p whose existence is claimed in the part 5) of the

proposition. To be more formal, we seek a sub-linebundle of $\mathcal{E} = \mathcal{N}(-3H)$ which coincides with $\mathcal{O}_{F_e}(-1)$ on every fibre F_e of π . The construction is the same as in the proof of part 1) of the proposition. Namely, we know that $\mathcal{N}(-2H) \otimes \mathcal{O}_{F_e} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Hence, the direct image $\pi_*(\mathcal{N}(-2H))$ is a line bundle on E which will be denoted $\mathcal{O}_E(D)$. This gives rise to the exact sequence

$$0 \longrightarrow \pi^*(\mathcal{O}_E(D)) \longrightarrow \mathcal{N}(-2H) \longrightarrow \mathcal{O}_X(K_X + H - \pi^*(D)) \longrightarrow 0$$

and hence, the sought after sub-linebundle of $\mathcal{N}(-3H)$ is $\mathcal{O}_X(-H + \pi^*D)$. Furthermore, the second Chern class computation from the above exact sequence yields $\deg(D) = -5$. In addition, by construction, the section $\gamma : X \longrightarrow Y$ corresponding to the subbundle $\mathcal{O}_X(-H + \pi^*D) \subset \mathcal{N}(-3H)$ has the property $\mathcal{O}_X(H - \pi^*D) = \gamma^*(\mathcal{O}_Y(1))$. Hence the image $X' = (\varphi \circ \gamma)(X)$ has degree

$$\deg(X') = (H - \pi^*D)^2 = (H + 5f)^2 = 5 + 10 = 15.$$

□

Remark. All the facts in Proposition A.7, with the possible exception of identifying the \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{E})$, have been proved in [4]. It seems to us that taking the vector bundle $\mathcal{E}^* = \mathcal{N}_X^*(3H)$ on the scroll X as the starting point makes its relation with the associated elliptic quintic curve more natural.

The property of $\mathbb{P}(\mathcal{E})$ being a fibration by Hirzebruch surfaces Σ_1 , which comes almost for free in our exposition, is all one needs to establish the relationship with the space of quadrics passing through the elliptic normal quintic curve $\varphi(\Delta_E)$ and eventually, show that $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{N}_{\varphi(\Delta_E)} \otimes \mathcal{O}_{\mathbb{P}(W^*)}(-2))$ are related by a birational morphism²³, where $\mathcal{N}_{\varphi(\Delta_E)}$ is the normal bundle of $\varphi(\Delta_E)$ in $\mathbb{P}(W^*)$. See [4] for details.

With the above description of the morphism $\varphi : Y \rightarrow \mathbb{P}(W^*)$ we can now explain how to go from φ back to the embedding $X \subset \mathbb{P}(V^*)$. For this, we use the following notation.

- $\langle z, z' \rangle$ is the line through the distinct points $z, z' \in \mathbb{P}^4$
- $Y_x = p^{-1}(x)$, the fibre of p over $x \in X$.

PROPOSITION A.9. *The isomorphism*

$$W = H^0(\mathcal{N}^*(3H)) = H^0(\mathcal{E}^*) \cong H^0(\mathcal{O}_Y(1)), \quad (\text{A.12})$$

establishes the following geometric correspondence between the hyperplanes in $\mathbb{P}(W^)$ and 0-cycles on X .*

Let M be a hyperplane in $\mathbb{P}(W^)$ intersecting the elliptic normal curve $\varphi(\Delta_E)$ along the divisor $D_M = M \cdot \varphi(\Delta_E)$ consisting of 5 distinct points²⁴. Then the configuration of the ten secant lines of $\varphi(\Delta_E)$*

$$\sum_{e \neq e' \in D_M} \langle e, e' \rangle,$$

gives rise to the 0-cycle

$$Z_M = \sum_{e \neq e' \in D_M} p_*(Y_{\tau(e, e')}) \quad (\text{A.13})$$

on X , where $\varphi(Y_{\tau(e, e')}) = \langle e, e' \rangle$. That 0-cycle is the scheme of zeros of a unique projective section $[s_M] \in \mathbb{P}(H^0(\mathcal{N}^(3H)))$ corresponding to M under the isomorphism (A.12).*

²³The morphism is the relative blow-down of $\mathbb{P}(\mathcal{E})$ along the image of the section γ in Proposition A.7, 5).

²⁴In what follows we abuse the notation by tacitly identifying $\varphi(\Delta_E)$ with E .

Proof. Let $[s_M] \in \mathbb{P}(H^0(\mathcal{N}^*(3H)))$ be the projective section corresponding to a hyperplane M . The zero locus $(s_M = 0) \subset X$ parametrizes the fibres of the projection $p : Y \rightarrow X$ which are mapped onto the secant lines of $\varphi(\Delta_E)$ contained in M and those are precisely the ones appearing on the right side of the equality (A.13). \square

We can now reconstruct²⁵ $X \subset \mathbb{P}(V^*)$ from the correspondence

$$\mathbb{P}(W) \ni M \mapsto Z_M = \sum_{e \neq e' \in D_M} p_*(Y_{\tau(e, e')})$$

described in Proposition A.9. The 0-cycle Z_M admits a distinguished decomposition into 5 subcycles

$$Z_M = \bigcup_{e \in D_M} Z_M^e \quad \text{where} \quad Z_M^e = \sum_{e' \in D_M \setminus \{e\}} p_*(Y_{\tau(e, e')}), \quad (\text{A.14})$$

i.e., Z_M^e parametrizes the fibres of $p : Y \rightarrow X$ mapped by φ onto the rulings of the cone $\varphi(\mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\Gamma_e}))$ that are contained in M . Hence we have

$$Z_M^e \subset \Gamma_e \quad \text{for every } e \in D_M.$$

The configuration of 5 curves $\{\Gamma_e\}_{e \in D_M}$, on the side of $\mathbb{P}(V^*)$, gives rise to 5 planes $\{\Pi_e\}_{e \in D_M}$, where Π_e is the span of Γ_e in the embedding $X \subset \mathbb{P}(V^*)$. Now one recovers the Segre cubic $V_3(s_M)$, the cubic hypersurface corresponding to s_M under the isomorphism $H^0(\mathcal{N}^*(3H)) \cong I_X(3)$ in Theorem 8.1, as the union of the lines in $\mathbb{P}(V^*)$ intersecting (any) four of the five planes of the collection $\{\Pi_e\}_{e \in D_M}$. The 0-cycle $Z_M = (s_M = 0) = \text{Sing}(V_3(s_M))$ is seen in $\mathbb{P}(V^*)$ as the cycle

$$Z_M = \sum_{e \neq e' \in D_M} \Pi_e \cap \Pi_{e'}.$$

composed of points of pairwise intersections of the planes $\{\Pi_e\}_{e \in D_M}$. As M varies in the complement of the dual variety of the elliptic normal curve $\varphi(\Delta_E)$, the zero cycles Z_M sweep out the scroll X .

Let U be the Zariski dense open subset of $\mathbb{P}(W)$ parametrizing regular global sections of $\mathcal{N}^*(3H)$. From the construction above U coincides with the set of hyperplanes in $\mathbb{P}(W^*)$ intersecting the elliptic normal quintic curve $\varphi(\Delta_E) \subset \mathbb{P}(W^*)$ along five distinct points. Thus the complement

$$T_{\varphi(\Delta_E)} := \mathbb{P}(W^*) \setminus U$$

is the dual variety of $\varphi(\Delta_E)$, i.e., the closed points of $T_{\varphi(\Delta_E)}$ parametrize the hyperplanes in $\mathbb{P}(W^*)$ which are tangent to $\varphi(\Delta_E)$ at some point. Furthermore, the construction described above and the identification

$$\mathbb{P}(W) \cong |I_X(3)| \quad (\text{A.15})$$

in Theorem 8.1, 3), imply that the Zariski open subset U , under the above identification, corresponds to Segre cubics in $\mathbb{P}(V^*)$ containing X . Thus the dual variety $T_{\varphi(\Delta_E)}$, via the identification (A.15), parametrizes cubic hypersurfaces in $|I_X(3)|$ with degenerate singular locus. To see this degeneracy we take a hyperplane M in $\mathbb{P}(V^*)$ which has a contact of order

²⁵We are grateful to I. Dolgachev for pointing out to us the construction that follows; according to him, it was Segre's way to see an elliptic scroll inside a Segre cubic.

2 with $\varphi(\Delta_E)$ at a point e_0 and is transversal to $\varphi(\Delta_E)$ at the remaining three points which we denote e_i , $i = 1, 2, 3$. Thus the divisor $D_M = M \cdot \varphi(\Delta_E)$ has the form

$$D_M = e_0 + t_{e_0} + e_1 + e_2 + e_3,$$

where t_{e_0} denotes a tangent vector of $\varphi(\Delta_E)$ at e_0 . The corresponding configuration of secant lines of $\varphi(\Delta_E)$ contained in M consists of:

- $\langle e_0, e_i \rangle$, $i = 1, 2, 3$, with multiplicity 2,
- the embedded tangent line $\langle e_0, t_{e_0} \rangle$ of $\varphi(\Delta_E)$ at e_0 ,
- three lines $\langle e_i, e_j \rangle$, $1 \leq i < j \leq 3$.

This accounts for ten secant lines counted with multiplicities. Since the scheme of zeros Z_M of the projective section $[s_M] \in \mathbb{P}(W)$ corresponding M under the isomorphism in (A.12) must parametrize the secants of $\varphi(\Delta_E)$ contained in M and $\deg(Z_M) = 10$, we deduce that no other secant of $\varphi(\Delta_E)$ is contained in M and the 0-cycle Z_M has the form

$$Z_M = 2 \sum_{i=1}^3 \tau(e_0, e_i) + \tau(e_0, e_0) + \sum_{1 \leq i < j \leq 3} \tau(e_i, e_j), \quad (\text{A.16})$$

where $\tau : E \times E \rightarrow X$ is the double covering in (A.5).

Now we move on the side of the embedding $X \subset \mathbb{P}(V^*)$. Denote by Γ_i the curves of the family $\{\Gamma_b\}_{b \in E}$ in (A.6) corresponding to the points e_i , with $i = 0, \dots, 3$, (we are making here the obvious identification of E and $\varphi(\Delta_E)$) and let $\{\Pi_i = \text{Span}(\Gamma_i)\}_{0 \leq i \leq 3}$ be the corresponding configuration of planes. By construction, each curve Γ_i is identified with the projection of $\varphi(\Delta_E)$ from the point e_i and comes along with a distinguished divisor D_i in the linear system $|\mathcal{O}_{\Gamma_i}(H - K_X)| = |\mathcal{O}_{\Gamma_i}(H + \Gamma_i)|$:

$$\begin{aligned} D_0 &= e_0 + e_1 + e_2 + e_3 \\ D_i &= e_0 + t_{e_0} + \sum_{j \neq i} e_j \quad \text{for } i = 1, 2, 3. \end{aligned}$$

In particular, $e_0 = \Gamma_0 \cdot \Gamma_0$ and $e_1 + e_2 + e_3 = D_0 - e_0 \sim H \cdot \Gamma_0$. This means that the plane cubic section $\Gamma_0 \subset \Pi_0$ comes along with the line $l_0 = \text{Span}\{e_1, e_2, e_3\}$ spanned by the three colinear points e_i ($i = 1, 2, 3$).

With the four distinct planes $\{\Pi_i\}_{0 \leq i \leq 3}$ the construction of Segre associates

$$V(e_0, l_0) := \text{the union of lines in } \mathbb{P}(V^*) \text{ intersecting the four planes } \{\Pi_i\}_{0 \leq i \leq 3}. \quad (\text{A.17})$$

This is a cubic hypersurface containing X and its singular locus $\text{Sing}(V(e_0, l_0))$, according to Theorem 8.1, 3), intersects X along the 0-cycle Z_M in (A.16). In particular, $V(e_0, l_0)$ is singular at three colinear points e_1, e_2, e_3 and hence, it must be singular along the line l_0 . This together with Remark 8.3 implies that the 1-dimensional part of $\text{Sing}(V(e_0, l_0))$ is the line l_0 . In addition, the formulas $Z_M = \text{Sing}(V(e_0, l_0)) \cdot X$ and (A.16) tell us that $V(e_0, l_0)$ is singular at four distinct points, $\tau(e_0, e_0)$, $\tau(e_1, e_2)$, $\tau(e_1, e_3)$ and $\tau(e_2, e_3)$, lying outside of l_0 . Since this is the maximal possible number of isolated singularities for a cubic hypersurface in \mathbb{P}^4 with precisely one line as the 1-dimensional part of its singular locus, we deduce

$$\text{Sing}(V(e_0, l_0)) = l_0 \cup \{\tau(e_0, e_0), \tau(e_1, e_2), \tau(e_1, e_3), \tau(e_2, e_3)\}.$$

By the above construction, a Zariski dense open subset of the dual variety $T_{\varphi(\Delta_E)}$ corresponds to the open part of ‘degenerate’ cubics in $|I_X(3)|$, *i.e.*, cubics having precisely a line as the 1-dimensional part of their singular locus. Since $T_{\varphi(\Delta_E)}$ is irreducible, we deduce the following.

PROPOSITION A.10. *Under the isomorphism $\mathbb{P}(W) \cong |I_X(3)|$ in (A.15), the dual variety $T_{\varphi(\Delta_E)}$ corresponds to the hypersurface C_1 in $|I_X(3)|$ parametrizing the cubic hypersurfaces containing X and having precisely a line as the 1-dimensional part of their singular locus. Furthermore, the singular lines of the cubics in C_1 are precisely the trisecant lines of $X \subset \mathbb{P}(V^*)$. In particular, $T_{\varphi(\Delta_E)}$ and C_1 are both isomorphic to the variety of trisecants of X .*

Remark. The desingularization of the dual variety $T_{\varphi(\Delta_E)}$ is the projectivization $\mathbb{P}(\mathcal{S})$ of $\mathcal{S} := N_{\varphi(\Delta_E)}^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}(1)$, the twisted conormal bundle of $\varphi(\Delta_E) \subset \mathbb{P}(W^*)$, while the desingularization of the variety of trisecants of $X \subset \mathbb{P}(V^*)$ is $\mathbb{P}(\mathcal{F})$, where \mathcal{F} is the bundle defined in Proposition A.4. So the last assertion of Proposition A.10 implies an isomorphism between $\mathbb{P}(\mathcal{S})$ and $\mathbb{P}(\mathcal{F})$, hence an identification $\mathcal{S} \cong \mathcal{F} \otimes \mathcal{L}$, for some line bundle \mathcal{L} on E . Comparing the degrees on both sides, we obtain $\deg(\mathcal{L}) = -5$. Thus

$$\mathcal{L} = \mathcal{O}_{\varphi(\Delta_E)}(-1) \otimes \mathcal{L}',$$

where $\mathcal{L}' \in \text{Pic}^0(E)$ and $\mathcal{O}_{\varphi(\Delta_E)}(1) = \mathcal{O}_{\mathbb{P}(W^*)}(1) \otimes \mathcal{O}_{\varphi(\Delta_E)}$. Thus, the above identification takes the form

$$N_{\varphi(\Delta_E)}^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}(2) \cong \mathcal{F} \otimes \mathcal{L}'.$$

This isomorphism can be viewed as the vector bundle version of the quadro-cubic Cremona transformation, see [4] for another proof of this isomorphism.

A.3. The decomposition (A.14) and the configuration $(10_4, 15_6)$ of Segre

It is well-known that the ten ordinary double points of a Segre cubic hypersurface in \mathbb{P}^4 have remarkable combinatorial properties: there are fifteen planes, each spanned precisely by four singular points, and through every singular point pass precisely six of the fifteen planes. This is called a $(10_4, 15_6)$ configuration. In this subsection we revisit this configuration in the light of the correspondence between the embedding $X \subset \mathbb{P}(V^*)$ and the geometry of the twisted conormal bundle $\mathcal{N}_X^*(3H)$ encapsulated in the properties of the morphism $\varphi : Y = \mathbb{P}(\mathcal{N}_X(-3H)) \rightarrow \mathbb{P}(W^*)$, see Proposition A.7. We keep the notation of the previous subsection unless stated otherwise.

On the side of the vector bundle $\mathcal{N}_X^*(3H)$ we take a global section s whose zero locus $Z_s = (s = 0)$ consists of ten distinct points on X . According to Theorem 8.1, the section s corresponds to a Segre cubic denoted $V_3(s) \in \mathbb{P}(I_X(3))$, and under this correspondence

$$\text{Sing}(V_3(s)) = Z_s,$$

where $\text{Sing}(V_3(s))$ stands for the singular locus of $V_3(s)$. So from the Segre’s works we know that Z_s is a $(10_4, 15_6)$ configuration. We wish to be more specific and identify the fifteen planes as well as the 4-point subcycles of Z_s spanning these planes in the light of geometry of the morphism φ and of the embedding $X \subset \mathbb{P}(V^*)$.

We know from the previous section that φ gives rise to a distinguished decomposition of Z_s into subcycles of degree 4. Namely, the isomorphism

$$H^0(\mathcal{N}^*(3H)) \cong H^0(\mathcal{O}_Y(1)) = W$$

identifies s with the hyperplane M_s in $\mathbb{P}(W^*)$. Denote by D_s the divisor cut out by M_s on the elliptic normal curve $\varphi(\Delta_E)$, i.e.,

$$D_s := M_s \bigcap \varphi(\Delta_E).$$

Then (A.14) provides the decomposition

$$Z_s = \bigcup_{e \in D_s} Z_s^e, \quad (\text{A.18})$$

where, to shorten the notation, we write Z_s^e instead of $Z_{M_s}^e$. We know that each Z_s^e is a subcycle of degree 4 in Z_s lying on the plane elliptic curve $\Gamma_e \subset \mathbb{P}(V^*)$ and no other point of Z_s is contained in Γ_e . In particular, the plane Π_e spanned by Γ_e contains Z_s^e and no other point of Z_s . Furthermore, since no three points of Z_s can be colinear²⁶, the plane Π_e is spanned by Z_s^e . This gives us a collection of five planes $\{\Pi_e\}_{e \in D_s}$ with the property that for every $e \neq e'$ the intersection

$$\Pi_e \bigcap \Pi_{e'} = \Gamma_e \cdot \Gamma_{e'}$$

is a single point, see the proof of Proposition A.5. Furthermore, the 0-cycle Z_s is seen in the embedding $X \subset \mathbb{P}(V^*)$ as the cycle of points in $\mathbb{P}(V^*)$ formed by the above pairwise intersections

$$Z_s = \sum_{e \neq e' \in D_s} \Pi_e \bigcap \Pi_{e'}. \quad (\text{A.19})$$

In the sequel we set

$$e \cdot e' := \Pi_e \bigcap \Pi_{e'} \quad \text{for } e \neq e' \in D_s. \quad (\text{A.20})$$

With this notation the formula (A.19) reads

$$Z_s = \sum_{e \neq e' \in D_s} e \cdot e',$$

while, for every $e \in D_s$, the subcycle Z_s^e in the decomposition (A.18) take the form

$$Z_s^e = \Pi_e \bigcap Z_s = \sum_{e' \neq e \in D_s} e \cdot e'.$$

We now give a description of the remaining ten planes of $(10_4, 15_6)$ configuration.

LEMMA A.11. *For every $e \neq e' \in D_s$, let*

$$Z_s^{e \cdot e'} = Z_s \setminus \left(Z_s^e \bigcup Z_s^{e'} \right) + e \cdot e'.$$

Then $Z_s^{e \cdot e'}$ is a 4-degree subcycle of Z_s that spans in $\mathbb{P}(V^)$ a plane denoted by $\Pi_{e \cdot e'}$.*

Proof. The fact that $Z_s^{e \cdot e'}$ consists of four points is obvious. To see that these points span a plane in $\mathbb{P}(V^*)$ we use the identification of Z_s with the singular locus $\text{Sing}(V_3(s))$ of the Segre cubic $V_3(s)$, see Theorem 8.1. We fix the point $e \cdot e' = \Pi_e \bigcap \Pi_{e'}$ and consider the projection from this point onto a complementary \mathbb{P}^3 which intersects $V_3(s)$ transversely

²⁶Otherwise the line through the three colinear points of Z_s is a singular line of the cubic $V_3(s)$ which is impossible.

along a smooth cubic surface, call it F . Then the remaining nine points of $Z_s = \text{Sing}(V_3(s))$ project to the nine nodes of a $(3, 3)$ -divisor

$$A = F \bigcap Q,$$

where Q is a smooth quadric in \mathbb{P}^3 , the image of the tangent cone of $V_3(s)$ at the point $e \cdot e'$. The nine nodes force the divisor A to be completely reducible, *i.e.*, A has the form

$$A = \sum_{i=1}^3 f_i + \sum_{i=1}^3 g_i,$$

where f_i (resp. g_i), for $i = 1, 2, 3$, are three disjoint rulings of Q such that the 0-cycle

$$Z'_s = \sum_{1 \leq i, j \leq 3} f_i \bigcap g_j$$

is the image of $Z_s \setminus \{e \cdot e'\}$ under the projection from $e \cdot e'$, see [12] for an excellent account of the above construction.

Under the projection from $e \cdot e'$, the planes Π_e and $\Pi_{e'}$ go to two skew rulings, say f_1 and f_2 . Hence the points Z_s^e and $Z_s^{e'}$ are mapped to the points $\sum_{j=1}^3 f_1 \bigcap g_j$ and $\sum_{j=1}^3 f_2 \bigcap g_j$ respectively. Therefore, the remaining points $Z_s \setminus (Z_s^e \cup Z_s^{e'})$ of $Z_s \setminus \{e \cdot e'\}$ must go to $\sum_{j=1}^3 f_3 \bigcap g_j$. Hence the three points $Z_s \setminus (Z_s^e \cup Z_s^{e'})$ together with $e \cdot e'$ lie in the plane spanned by the line f_3 and the point $e \cdot e'$. Using again the fact that no three points of Z_s are colinear, we deduce the assertion of the claim. \square

The five planes $\{\Pi_e\}_{e \in D_s}$ together with the planes $\Pi_{e \cdot e'}$, ($e \neq e' \in D_s$), in Claim A.11 account for fifteen planes in the $(10_6, 15_4)$ configuration. This collection of planes will be denoted by

$$\mathfrak{P}_s := \{\Pi_e, \Pi_{e \cdot e'} \mid e \in D_s, e \neq e' \in D_s\}. \quad (\text{A.21})$$

The following is obvious from the construction.

LEMMA A.12. *Every point $e \cdot e' \in Z_s$ lies precisely on the following six planes of the collection \mathfrak{P}_s ,*

$$\mathfrak{P}_s^{e \cdot e'} = \{\Pi_e, \Pi_{e'}, \Pi_{e \cdot e'}, \Pi_{e'' \cdot e'''} \mid e'' \neq e''' \in D_s \setminus \{e, e'\}\}.$$

Furthermore, each subset of the partition $\mathfrak{P}_s^{e \cdot e'} = {}^1\mathfrak{P}_s^{e \cdot e'} \cup {}^2\mathfrak{P}_s^{e \cdot e'}$, where

$${}^1\mathfrak{P}_s^{e \cdot e'} = \{\Pi_e, \Pi_{e'}, \Pi_{e \cdot e'}\} \quad \text{and} \quad {}^2\mathfrak{P}_s^{e \cdot e'} = \{\Pi_{e'' \cdot e'''} \mid e'' \neq e''' \in D_s \setminus \{e, e'\}\},$$

consists of three planes of $\mathfrak{P}_s^{e \cdot e'}$ which intersect precisely at $e \cdot e'$, while the planes taken from different subsets intersect along a line. More precisely, if $D_s = \{e, e', e'', e''', c\}$, then

$$\Pi_e \cap \Pi_{e'' \cdot e'''} = \langle e \cdot e', e \cdot c \rangle, \quad \Pi_{e'} \bigcap \Pi_{e'' \cdot e'''} = \langle e \cdot e', e' \cdot c \rangle, \quad \Pi_{e \cdot e'} \bigcap \Pi_{e'' \cdot e'''} = \langle e \cdot e', e'' \cdot e''' \rangle,$$

where for two distinct points x, y in a projective space, $\langle x, y \rangle$ denotes the line spanned by those points.

Remark A.13. The above two claims give a precise recipe of how to recover the collection \mathfrak{P}_s of the fifteen planes of the $(10_4, 15_6)$ configuration from the cycle Z_s of ten points on $X \subset V_3(s)$. It should be pointed out that the collection \mathfrak{P}_s has an extra feature of being ‘polarized’ into two types of planes: Π_e , $e \in D_s$, and $\Pi_{e \cdot e'}$, for $e \neq e' \in D_s$. The presence of this polarization is due of course to the fact that Z_s is not just the singular locus of a Segre cubic $V_3(s)$, but also lies on the scroll X inside of $V_3(s)$. To be even more precise, the above polarization emerges from the fact that Z_s is the zero locus of a section of a vector bundle on X .

The two types of planes in \mathfrak{P}_s play different roles with respect to the embedding $X \subset \mathbb{P}(V^*)$. By construction, the planes $\{\Pi_e\}_{e \in D_s}$ are distinguished by the property that the intersection $X \cap \Pi_e$ is the plane elliptic curve Γ_e , for every $e \in D_s$. The following lemma gives a similar characterization of the planes $\Pi_{e \cdot e'}$.

LEMMA A.14. *For every pair $e \neq e' \in D_s$, the plane $\Pi_{e \cdot e'}$ intersects the scroll X along the subcycle $Z_s^{e \cdot e'}$ and the ruling $l_{e \cdot e'}$ of X passing through the point $e \cdot e'$.*

Proof. Let $\{c, c', c''\}$ be the complement $D_s \setminus \{e, e'\}$. From Claim A.11 it follows that the subcycle

$$Z_s^{e \cdot e'} = c \cdot c' + c \cdot c'' + c' \cdot c'' + e \cdot e'$$

is contained in the intersection $X \cap \Pi_{e \cdot e'}$. Observe that the line $L = \langle c \cdot c', c \cdot c'' \rangle$ is contained in the plane Π_c and hence it intersects the curve Γ_c along three points (the degree 3 divisor)

$$T = L \cdot \Gamma_c = c \cdot c' + c \cdot c'' + t.$$

The same holds for the line $L' = \langle c \cdot c', c' \cdot c'' \rangle$ (resp. $L'' = \langle c \cdot c'', c' \cdot c'' \rangle$) and leads to

$$T' = L' \cdot \Gamma_c = c \cdot c' + c' \cdot c'' + t' \quad (\text{resp. } T'' = c \cdot c'' + c' \cdot c'' + t'').$$

It follows that the plane $\Pi_{e \cdot e'}$ intersects X along seven points, $Z_s^{e \cdot e'} + t + t' + t''$. Since the degree of X is 5, we deduce that the intersection of $\Pi_{e \cdot e'}$ and X is not proper, *i.e.*, that the intersection has a 1-dimensional component, call it F . The scheme F can be either a plane cubic or a ruling of X . According to Proposition A.2, the first possibility means that F is one of the curves $\{\Gamma_b\}_{b \in E}$ and this is clearly impossible. Hence F is a ruling of X .

To identify this ruling we observe that it must meet all curves $\{\Gamma_b\}_{b \in E}$. In particular, it must intersect Γ_e . Hence F must pass through the intersection $\Pi_{e \cdot e'} \cap \Pi_e$ and this, in view of Lemma A.12, is the point $e \cdot e'$. Hence F is the ruling of X passing through the point $e \cdot e'$ as asserted. \square

A.4. Toward a categorification of the configuration $(10_4, 15_6)$ of Segre

Conceptually, the whole approach of our paper can be termed as a representation of various geometric or cohomological entities attached to a surface in \mathbb{P}^4 in the category of complexes of coherent sheaves on that surface.

In this subsection we apply this approach to the $(10_4, 15_6)$ configuration of Segre considered in the previous section. Namely, with our geometric set up of an elliptic scroll X embedded in $\mathbb{P}(V^*)$, we have seen how the scheme of zeros Z_s of a regular²⁷ global section s

²⁷A *regular* global section is a global section with simple isolated zeros.

of $\mathcal{N}_X^*(3H)$ acquires the structure of the $(10_4, 15_6)$ configuration of Segre. With our notation and results from the previous subsection ,

$$Z_s = \sum_{e \neq e' \in D_s} e \cdot e',$$

where D_s is intrinsically defined by s ; it is a set of five distinct points on the elliptic curve E , the base of X . See (A.19) and (A.20) for notation. We have noticed that Z_s contains the distinguished subcycles of degree 4,

$$Z_s^e = \sum_{e' \in D_s \setminus \{e\}} e \cdot e', \quad Z_s^{e \cdot e'} = Z_s \setminus \left(Z_s^e \cup Z_s^{e'} \right) + e \cdot e', \quad (\text{A.22})$$

which have the geometric property of spanning the planes Π_e and $\Pi_{e \cdot e'}$ in $\mathbb{P}(V^*)$. These planes form the collection \mathfrak{P}_s of fifteen planes in (A.21). We suggest that there is a lifting of (Z_s, \mathfrak{P}_s) to the category $\mathbf{Comp}(X)$ of (short) exact complexes of torsion free sheaves on X and hence, to the derived category $\mathfrak{D}(X)$ of the coherent sheaves on X . Before we go on, let us be more precise about our suggestion.

The first step of a categorification process is to assign a complex to every subcycle in (A.22). The second one is to turn (Z_s, \mathfrak{P}_s) into a category and then to check that the morphisms of that category go to morphisms of complexes.

The main result of this subsection is a realization of the first step. As for the second, let us just indicate here how one could think of (Z_s, \mathfrak{P}_s) as a category. This can be achieved by turning \mathfrak{P}_s into a graph, call it $\mathcal{C}(\mathfrak{P}_s)$:

- the vertices of $\mathcal{C}(\mathfrak{P}_s)$ are the planes of the collection \mathfrak{P}_s ,
- there is an edge between two vertices if and only if the corresponding planes intersect along a line.

Of course, the graph $\mathcal{C}(\mathfrak{P}_s)$ is obviously a category: the objects are the vertices of $\mathcal{C}(\mathfrak{P}_s)$ and the morphisms between two objects, say from Π to Π' , are the paths, composed of edges of the graph, beginning at Π and ending at Π' . With this understood, we define $\mathcal{C}(\mathfrak{P}_s)$ to be the category of the Segre configuration (Z_s, \mathfrak{P}_s) and we propose that there should be geometrically interesting functor(s)

$$\mathfrak{F} : \mathcal{C}(\mathfrak{P}_s) \longrightarrow \mathbf{Comp}(X) \quad (\text{resp. } \mathfrak{D}(X)). \quad (\text{A.23})$$

In the sequel, we construct such a functor \mathfrak{F} on the level of objects. This is essentially Serre construction and is based on the following observation.

LEMMA A.15. *Let Z be one of the subcycles of degree 4 in (A.22). Then there is an extension sequence*

$$0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{F}_Z \longrightarrow \mathcal{I}_Z(H) \longrightarrow 0 \quad (\text{A.24})$$

intrinsically attached to Z , where \mathcal{I}_Z is the ideal sheaf of Z in X and \mathcal{F}_Z is a locally free sheaf of rank 2 with Chern invariants

$$c_1(\mathcal{F}_Z) = K_X + H \quad \text{and} \quad c_2(\mathcal{F}_Z) = -1.$$

Proof. The geometric condition of Z spanning a plane in $\mathbb{P}(V^*)$ is translated, via the exact sequence

$$0 \longrightarrow \mathcal{J}_Z(H) \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{O}_Z(H) \longrightarrow 0,$$

to the cohomological condition $h^1(\mathcal{J}_Z(H)) = 1$. This and the Serre duality

$$H^1(\mathcal{J}_Z(H))^* \cong \text{Ext}^1(\mathcal{J}_Z(H), \mathcal{O}_X(K_X)).$$

imply that there is an extension as in (A.24) and such an extension is unique, up to the \mathbb{C}^\times -action of scaling the morphisms in that sequence. Furthermore, since for any proper subscheme $Z' \subset Z$ the cohomology $H^1(\mathcal{J}_{Z'}(H)) = 0$, it follows by a lemma of Serre, [27, Lemma 5.1.2], that the sheaf \mathcal{F}_Z in (A.24) is locally free. Its invariants are immediately deduced from (A.24). \square

Next we investigate the vector bundle \mathcal{F}_Z in (A.24).

LEMMA A.16. *The vector bundle \mathcal{F}_Z in (A.24) is H -unstable. More precisely, there is an effective nonzero divisor A_Z on X such that \mathcal{F}_Z fits into the short exact sequence*

$$0 \longrightarrow \mathcal{O}_X(A_Z) \longrightarrow \mathcal{F}_Z \longrightarrow \mathcal{J}_{Z'}(K_X + H - A_Z) \longrightarrow 0, \quad (\text{A.25})$$

where Z' is a 0-dimensional subscheme of X and $\mathcal{J}_{Z'}$ is its ideal sheaf. Furthermore, $\mathcal{O}_X(A_Z)$ is the H -maximal destabilizing subsheaf of \mathcal{F}_Z .

Proof. From (A.24) it follows that

$$h^0(\mathcal{F}_Z) \geq h^0(\mathcal{J}_Z(H)) - h^1(\mathcal{O}_X(K_X)) = 2 - 1 = 1.$$

Hence \mathcal{F}_Z has a nonzero global section, call it f . This and the Chern invariant $c_2(\mathcal{F}_Z) = -1$ computed in Lemma A.15, imply that the subscheme of zeros of f must have a divisorial part which is the divisor A_Z of the lemma. Hence $\mathcal{F}_Z(-A_Z)$ has a nonzero global section f' whose zero locus is 0-dimensional. The asserted sequence (A.25) is the Koszul sequence of f' tensored with $\mathcal{O}_X(A_Z)$.

From $H \cdot A_Z > 0 = H \cdot (K_X + H) = H \cdot c_1(\mathcal{F}_Z)$ it follows that $\mathcal{O}_X(A_Z)$ is H -destabilizing. This together with the fact that the quotient sheaf in (A.25) is torsion free insures the maximality of $\mathcal{O}_X(A_Z)$. \square

Next we show how the destabilizing sequence (A.25) distinguishes between the two types of subcycles in (A.22). For this we put the defining extension sequence (A.24) together with the destabilizing one to obtain the diagram.

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathcal{O}_X(A_Z) & & & \\
 & & & \downarrow & \searrow & & \\
 0 & \longrightarrow & \mathcal{O}_X(K_X) & \longrightarrow & \mathcal{F}_Z & \longrightarrow & \mathcal{J}_Z(H) \longrightarrow 0 \\
 & & \searrow & & \downarrow & & \\
 & & & & \mathcal{J}_{Z'}(K_X + H - A_Z) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array} \quad (\text{A.26})$$

The morphisms defined by the slanted arrows are nonzero and hence give rise to a nonzero effective divisor $B_Z \in |\mathcal{J}_Z(H - A_Z)|$ (resp. $|\mathcal{J}_{Z'}(H - A_Z)|$). In particular, we obtain the decomposition

$$H = A_Z + B_Z$$

and claim the following.

LEMMA A.17. *Let $\pi : X \rightarrow E$ be the structure projection of X onto the elliptic curve E . Then $A_Z = \pi^*(\mathfrak{a})$, where \mathfrak{a} is a divisor of degree either 1 or 2 on E .*

Proof. Observe that $h^0(\mathcal{O}_X(A_Z)) \leq 2$, since otherwise

$$3 \leq h^0(\mathcal{O}_X(A_Z)) = h^0(\mathcal{O}_X(H - B_Z))$$

implies that B_Z is a line containing Z which is impossible since Z spans a plane. On the other hand, the Riemann-Roch for $\mathcal{O}_X(A_Z)$ gives

$$h^0(\mathcal{O}_X(A_Z)) = \frac{1}{2}(A_Z^2 - A_Z \cdot K_X) + h^1(\mathcal{O}_X(A_Z)). \quad (\text{A.27})$$

From the vertical sequence in (A.26) we obtain

$$-1 = c_2(\mathcal{F}_Z) = A_Z \cdot (K_X + H - A_Z) + \deg(Z').$$

This together with (A.27) implies

$$h^0(\mathcal{O}_X(A_Z)) = \frac{1}{2}(A_Z \cdot H + 1 + \deg(Z')) + h^1(\mathcal{O}_X(A_Z)).$$

This identity and the above upper bound $h^0(\mathcal{O}_X(A_Z)) \leq 2$ give the following possibilities:

- 1) $h^0(\mathcal{O}_X(A_Z)) = 1$ and $A_Z \cdot H = 1$, $\deg(Z') = 0$,
- 2) $h^0(\mathcal{O}_X(A_Z)) = 2$ and $A_Z \cdot H = 2$, $\deg(Z') = 1$,
- 3) $h^0(\mathcal{O}_X(A_Z)) = 2$ and $A_Z \cdot H = 3$, $\deg(Z') = 0$.

The third one can not hold, since it implies that B_Z must consist of two rulings of X that contain Z . This forces the two rulings to be contained in the plane spanned by Z .

The possibility 1) (resp. 2)) implies that A_Z is a (resp. the union of two) ruling(s) of X . Hence the assertion of the lemma. \square

Before we proceed, let us recall that the embedding $X \subset \mathbb{P}(V^*)$ is defined by $\mathcal{O}_X(H)$, where

$$H = \Gamma_o + \pi^*(D), \quad (\text{A.28})$$

with o and D being respectively a point and a divisor of degree 2 on E . With this in mind, we can now identify all the ingredients involved in the diagram (A.26) for each type of subcycle in (A.22).

PROPOSITION A.18. *Let Z be one of the degree 4 subcycles of Z_s appearing in (A.22).*

- 1) *If $Z = Z_s^e$, $e \in D_s$, then the destabilizing sequence (A.25) has the form*

$$0 \longrightarrow \mathcal{O}_X(\pi^*(\mathfrak{a}_e)) \longrightarrow \mathcal{F}_{Z_s^e} \longrightarrow \mathcal{J}_{z_e}(K_X + \Gamma_e) \longrightarrow 0,$$

where z_e is the point $\Gamma_e \cdot \Gamma_e$ and \mathfrak{a}_e is a divisor of degree 2 on E determined by the linear equivalence

$$\mathfrak{a}_e \sim D + o - e$$

with D as in (A.28).

2) If $Z = Z_s^{e \cdot e'}$, $e \neq e' \in D_s$, then the destabilizing sequence (A.25) has the form

$$0 \longrightarrow \mathcal{O}_X(l_x) \longrightarrow \mathcal{F}_{Z_s^{e \cdot e'}} \longrightarrow \mathcal{O}_X(K_X + H - l_x) \longrightarrow 0,$$

where l_x is the ruling of X passing through a point $x \in Z_s^{e \cdot e'}$. Furthermore, there is a unique divisor $R_{e \cdot e'} \in |H - l_x|$ passing through $Z_s^{e \cdot e'}$ and subject to one of the following properties:

- either $e \cdot e'$ is a unique point of $Z_s^{e \cdot e'}$ lying on the ruling $l_{e \cdot e'}$ and then $x = e \cdot e'$ and $R_{e \cdot e'}$ is a smooth elliptic curve of degree 4 and $Z_s^{e \cdot e'}$ is its plane section,
- or $x \neq e \cdot e'$, then the divisor $R_{e \cdot e'} = l_{e \cdot e'} + \Gamma_c$, for some $c \in D_s \setminus \{e, e'\}$, the ruling $l_{e \cdot e'}$ passes through two points $e \cdot e', c' \cdot c''$ of $Z_s^{e \cdot e'}$, where $\{c', c''\} = D_s \setminus \{e, e', c\}$, and the point $x = c \cdot c'$ or $c \cdot c''$.

Proof. To prove 1), set $A_e := H - \Gamma_e$ and observe that it has the form $A_e = \pi^*(\mathbf{a}_e)$, where \mathbf{a}_e is a divisor of degree 2 on E . Since the subcycle $Z = Z_s^e$ lies on the curve Γ_e , we have

$$H^0(\mathcal{J}_{Z_s^e}(H - \pi^*(\mathbf{a}_e))) = H^0(\mathcal{J}_{Z_s^e}(\Gamma_e)) \neq 0.$$

On the other hand, the defining sequence (A.24) for $Z = Z_s^e$, tensored with $\mathcal{O}_X(-\pi^*(\mathbf{a}_e))$, gives

$$0 \longrightarrow H^0(\mathcal{F}_{Z_s^e}(-\pi^*(\mathbf{a}_e))) \longrightarrow H^0(\mathcal{J}_{Z_s^e}(\Gamma_e)) \longrightarrow H^1(\mathcal{O}_X(K_X - \pi^*(\mathbf{a}_e))).$$

Since by Serre duality $H^1(\mathcal{O}_X(K_X - \pi^*(\mathbf{a}_e))) = H^1(\mathcal{O}_X(\pi^*(\mathbf{a}_e)))^* = 0$, we deduce

$$H^0(\mathcal{F}_{Z_s^e}(-\pi^*(\mathbf{a}_e))) \cong H^0(\mathcal{J}_{Z_s^e}(\Gamma_e)) \cong \mathbb{C}.$$

Hence $\mathcal{F}_{Z_s^e}(-\pi^*(\mathbf{a}_e))$ has, up to a nonzero scalar multiple, a unique nonzero global section. Furthermore, the scheme of zeros of this section is obviously 0-dimensional. Its Koszul sequence tensored with $\mathcal{O}_X(\pi^*(\mathbf{a}_e))$ gives

$$0 \longrightarrow \mathcal{O}_X(\pi^*(\mathbf{a}_e)) \longrightarrow \mathcal{F}_{Z_s^e} \longrightarrow \mathcal{J}_{Z'}(K_X + \Gamma_e) \longrightarrow 0,$$

where Z' is a single point, see the proof of Lemma A.17. Call this point z_e . From (A.26) it follows that $z_e \in \Gamma_e$. To identify it, we restrict the diagram (A.26) to the curve Γ_e and obtain the identity

$$\mathcal{O}_{\Gamma_e}(\mathbf{a}_e + z_e) = \mathcal{O}_X(K_X) \otimes \mathcal{O}_{\Gamma_e}(Z_s^e), \quad (\text{A.29})$$

where we tacitly use the identification of Γ_e with E . Furthermore, we have

$$\mathcal{O}_{\Gamma_e}(\mathbf{a}_e) = \mathcal{O}_{\Gamma_e}(H - \Gamma_e) \quad \text{and} \quad \mathcal{O}_{\Gamma_e}(Z_s^e) = \mathcal{O}_{\Gamma_e}(H - K_X),$$

where the first equality is the definition of the divisor \mathbf{a}_e and the second comes from realizing Z_s^e as the complete intersection of Γ_e with a smooth curve in $|H - K_X|$ containing Z_s . Substituting into (A.29), we obtain

$$\mathcal{O}_{\Gamma_e}(z_e) \otimes \mathcal{O}_{\Gamma_e}(-\Gamma_e) = \mathcal{O}_{\Gamma_e}$$

or, equivalently, $z_e \sim \Gamma_e \cdot \Gamma_e$. This together with $h^0(\mathcal{O}_{\Gamma_e}(z_e)) = 1$ imply the equality $z_e = \Gamma_e \cdot \Gamma_e$.

The linear equivalence asserted in 1) of the proposition follows from writing

$$\pi^*(\mathbf{a}_e) + \Gamma_e \sim H \sim \Gamma_o + \pi^*(D),$$

where the last equivalence is (A.28). Hence $\mathfrak{a}_e \sim D + o - e$ as divisors of E .

We turn now to the part 2) of the proposition. We know that the subcycle $Z_s^{e \cdot e'}$ does not lie on any of the curves of the family $\{\Gamma_b\}_{b \in E}$. This together with the first part of the proof and Lemma A.17 implies that the destabilizing sequence (A.25) for $\mathcal{F}_{Z_s^{e \cdot e'}}$ has the form

$$0 \longrightarrow \mathcal{O}_X(l_x) \longrightarrow \mathcal{F}_{Z_s^{e \cdot e'}} \longrightarrow \mathcal{O}_X(K_X + H - l_x) \longrightarrow 0,$$

where l_x is the ruling passing through some point $x \in X$. The slanted arrow in the upper right corner of (A.26) tells us that there is an effective divisor, call it $R_{e \cdot e'}$, in $|H - l_x|$ passing through $Z_s^{e \cdot e'}$. The uniqueness of this divisor follows from $h^0(\mathcal{J}_{Z_s^{e \cdot e'}}(H - l_x)) = h^0(\mathcal{F}_{Z_s^{e \cdot e'}}(-l_x)) = 1$, where the first equality comes from the horizontal sequence in (A.26) and the second one from the destabilizing sequence above.

It remains to analyse the properties of the divisor $R_{e \cdot e'}$ as well as the position of the point x . From the previous paragraph, we know already that $R_{e \cdot e'}$ is a unique effective divisor in $|H - l_x|$ passing through $Z_s^{e \cdot e'}$. From Lemma A.14 we also know that any divisor in $|H|$ passing through $Z_s^{e \cdot e'}$ must also contain the ruling $l_{e \cdot e'}$. Hence, if $l_x \neq l_{e \cdot e'}$, then the divisor $R_{e \cdot e'}$ has the form

$$R_{e \cdot e'} = l_{e \cdot e'} + \Gamma_b,$$

for some $b \in E$, and it must contain $Z_s^{e \cdot e'}$. But $l_{e \cdot e'}$ is allowed to contain at most two points of $Z_s^{e \cdot e'}$, since no three points in $Z_s^{e \cdot e'}$ are colinear. Therefore, Γ_b must contain a subscheme $Z_{\Gamma_b} \subset Z_s^{e \cdot e'}$ consisting of at least two points. Restricting now the diagram (A.26) to Γ_b , we obtain

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{O}_{\Gamma_b}(l_x) & & & & \\ & & \downarrow & \searrow & & & \\ 0 \longrightarrow & \mathcal{O}_X(K_X) \otimes \mathcal{O}_{\Gamma_b}(Z_{\Gamma_b}) & \longrightarrow & \mathcal{F}_Z \otimes \mathcal{O}_{\Gamma_b} & \longrightarrow & \mathcal{O}_X(H) \otimes \mathcal{O}_{\Gamma_b}(-Z_{\Gamma_b}) & \longrightarrow 0 \\ & \searrow & & \downarrow & & & \\ & & & \mathcal{J}_{Z'}(K_X + H - A_Z) & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

where the slanted arrows now are zero morphisms. Hence

$$\mathcal{O}_{\Gamma_b}(l_x) = \mathcal{O}_X(K_X) \otimes \mathcal{O}_{\Gamma_b}(Z_{\Gamma_b}).$$

This implies

$$l_x|_{\Gamma_b} \sim Z_{\Gamma_b} + K_X|_{\Gamma_b} = Z_{\Gamma_b} - \Gamma_b|_{\Gamma_b}.$$

In particular, $\deg(Z_{\Gamma_b}) = 2$. Hence, the remaining two point of $Z^{e \cdot e'}$ must lie on $l_{e \cdot e'}$.

To go further, recall that

$$Z^{e \cdot e'} = e \cdot e' + c \cdot c' + c \cdot c'' + c' \cdot c'',$$

where $\{c, c', c''\} = D_s \setminus \{e, e'\}$. Assume now that the ruling $l_{e \cdot e'}$ passes through $c' \cdot c''$ (in addition to the point $e \cdot e'$). Then Γ_b passes through $c \cdot c'$ and $c \cdot c''$ and hence Γ_b must be Γ_c .

The curve Γ_c intersects the plane $\Pi_{e \cdot e'}$, the span of $Z^{e \cdot e'}$, along the points situated on the line $\langle c \cdot c', c \cdot c'' \rangle$, the span of $c \cdot c'$ and $c \cdot c''$. Namely, we have

$$\Gamma_c \cap \Pi_{e \cdot e'} = \Gamma_c \cap \langle c \cdot c', c \cdot c'' \rangle = c \cdot c' + c \cdot c'' + l_{e \cdot e'} \cdot \Gamma_c.$$

Since the ruling l_x must pass through one of these points, we deduce that x is either $c \cdot c'$ or $c \cdot c''$ as asserted.

We now assume that $e \cdot e'$ is the only point of $Z_s^{e \cdot e'}$ lying on the ruling $l_{e \cdot e'}$. Then the preceding argument tells us that $l_x = l_{e \cdot e'}$ and $R_{e \cdot e'} \in |H - l_{e \cdot e'}|$ is a unique divisor passing through $Z_s^{e \cdot e'}$. Let us assume it to be reducible. Then it has the form

$$R_{e \cdot e'} = l_y + \Gamma_b.$$

Running the argument involving the previous diagram, we deduce that the ruling l_y must contain two points of $Z_s^{e \cdot e'}$. Hence l_y is contained in the plane $\Pi_{e \cdot e'}$ and then it coincides with $l_{e \cdot e'}$. This contradicts the assumption that $l_{e \cdot e'}$ contains only one point of the subcycle $Z_s^{e \cdot e'}$. Hence $R_{e \cdot e'}$ is irreducible and is a smooth section of $\pi : X \rightarrow E$. Its degree $H \cdot R_{e \cdot e'} = H \cdot (H - l_{e \cdot e'}) = 4$. Since the intersection $R_{e \cdot e'} \cap \Pi_{e \cdot e'} \supset Z_s^{e \cdot e'}$, we deduce the equality

$$R_{e \cdot e'} \cap \Pi_{e \cdot e'} = Z_s^{e \cdot e'}.$$

□

Now we define a functor \mathfrak{F} in (A.23) on the level of objects by

$$\begin{aligned} \mathfrak{F}(\Pi_e) &= \{0 \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{F}_{Z_s^e} \rightarrow \mathcal{J}_{Z_s^e}(H) \rightarrow 0\}, \quad \text{for every } e \in D_s, \\ \mathfrak{F}(\Pi_{e \cdot e'}) &= \{0 \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{F}_{Z_s^{e \cdot e'}} \rightarrow \mathcal{J}_{Z_s^{e \cdot e'}}(H) \rightarrow 0\}, \quad \text{for every } e \neq e' \in D_s. \end{aligned}$$

The further study of \mathfrak{F} and related topics will be considered elsewhere.

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